$$T_j^n = \frac{u_j^{n+1} - \dfrac{u_{j+1}^n + u_{j-1}^n}{2}}{\Delta t} + A_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$\approx \left[u_t + \frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx} + O(\Delta t^2) + O(\frac{\Delta x^4}{\Delta t}) \right]_j^n$$

$$+ \left[A(u_x + \frac{1}{6}\Delta x^2 u_{xxx}) + O(\Delta x^4) \right]_j^n$$

$$= [u_t + Au_x]_j^n + \left[\frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx} + \frac{\Delta x^2}{6} A u_{xxx} \right]_j^n$$

$$+ O(\Delta x^4 + \Delta t^{-1}\Delta x^4 + \Delta t^2).$$

Since u is exact solution, we get

$$[u_t + Au_x]_j^n = 0$$

hence

$$T_j^n = \left[\frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx} + \frac{\Delta x^2}{6} A u_{xxx} \right]_j^n$$

$$+ O(\Delta x^4 + \Delta t^{-1}\Delta x^4 + \Delta t^2).$$

which shows that the numerical scheme is consistent of order 2 in space and of order 1 in time as long as $\Delta t^{-1}\Delta x^2 \to 0$.

3.4 Convergence of the scheme

Definition: A finite difference scheme (Numerical method) is said to be convergent if for any fixed point (x^*, t^*) in a given domain $(0, X) \times (0, t_n)$,

$$x_j \to x^*, \;\; t_n \to t^* \quad \Rightarrow \quad U_j^n \to u(x^*, t^*)$$

the error in the approximation is given by

$$e_j^n = U_j^n - u(x_j, t_n).$$

Now U_j^n satisfies the finite difference scheme (7) exactly, while $u(x_j, t_n)$ leaves the remainder $T_j^n \Delta t$. Therefore the error is given by

$$e_j^{n+1} = \frac{1}{2}\left(1 - A_j^n \frac{\Delta t}{\Delta x}\right) e_{j+1}^n + \frac{1}{2}\left(1 + A_j^n \frac{\Delta t}{\Delta x}\right) e_{j-1}^n - \Delta t T_j^n$$

and $e_0^n = 0$.

Let $E^n = \max\left\{ \left| e_j^n \right|, \; j = 0, 1, ..., J \right\}$

Hence for $\left| A_j^n \frac{\Delta t}{\Delta x} \right| \le 1,$

$$E^{n+1} = \max_j \left| e_j^{n+1} \right| \le E^n + \Delta t \max_j \left| T_j^n \right| \quad \text{and } E^0 = 0$$

If we suppose that the truncation error is bounded i.e. $\left| T_j^n \right| \le T_{\max}$, then by induction method

$$E^n \le +n\Delta t T_{\max} \le t_n T_{\max},$$

which shows that the method has first-order convergent provided that the solution has bounded derivatives up to second order.

4. Numerical Experiments

In this section, we present some numerical examples to validate the predicted results established in the paper. We perform numerical computations using MATLAB. The maximum absolute errors for the considered examples are calculated using half mesh principle as the exact solution for the considered examples are not available [3]. We calculate the errors by refining the grid points. The error in the numerical approximation is given by

$$E(\Delta x, \Delta t) = \max_{0 \le j \le J, 0 \le n \le Nt} \left| U_{\Delta x}^{\Delta t}(j, n) - U_{\Delta x/2}^{\Delta t/2}(2j, 2n) \right|$$

In the following examples the domain of consideration is $\Omega = [0,1] \times [0, 0.7]$.

Example1. Consider the problem (2) with the following coefficients and initial- boundary conditions:

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}; \; b(x,t) = 5; \;\; c(x,t) = 10;$$

$$u(x,0) = \exp[-10(4x-1)^2]; \quad . \; u(s,t) = 0, \quad \forall s \in [-\alpha, 0].$$

Example2. Consider the problem (2) with the following coefficients and initial- boundary conditions:

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}; \quad b(x,t) = 1 + 2x^2t^2 + x^4;$$

$$c(x,t) = 1 + xt;$$

$$u(x,0) = \exp[-10(4x-1)^2]; \quad . \; u(s,t) = 0, \quad \forall s \in [-\alpha, 0] \; .$$

Example3. Consider the problem (2) with the following coefficients and initial- boundary conditions:

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}; \quad b(x,t) = \frac{1}{1+2x^2t^2+x^4};$$

$$c(x,t) = \frac{1}{1+xt};$$

$$u(x,0) = \exp[-10(4x-1)^2]; \quad . \; u(s,t) = 0, \quad \forall s \in [-\alpha, 0]$$

Table 1: The maximum absolute error for example 1

$\Delta t \downarrow \; \Delta x \to$	1/100	1/200	1/400	1/800
$\Delta x/2$	0.0588	0.0296	0.0149	0.0071
$\Delta x/4$	0.0294	0.0146	0.0069	0.0035
$\Delta x/8$	0.0145	0.0068	0.0034	0.0017
$\Delta x/16$	0.0067	0.0033	0.0016	0.0008

Table 2: The maximum absolute error for example 2

$\Delta t \downarrow \; \Delta x \to$	1/100	1/200	1/400	1/800
$\Delta x/2$	0.0582	0.0294	0.0147	0.0071
$\Delta x/4$	0.0292	0.0144	0.0068	0.0034
$\Delta x/8$	0.0141	0.0067	0.0033	0.0016
$\Delta x/16$	0.0065	0.0031	0.0015	0.0007

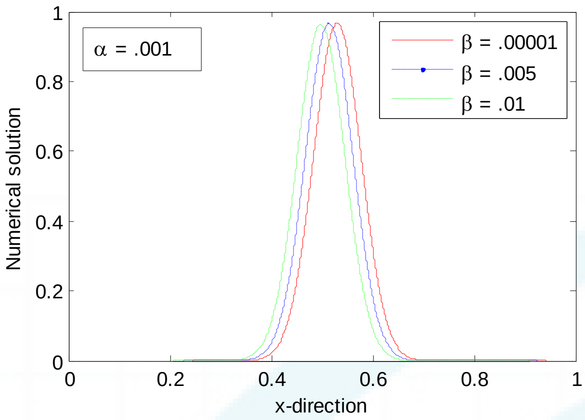

Figure1: The numerical solution for Example 1 at $t = 0.5$.

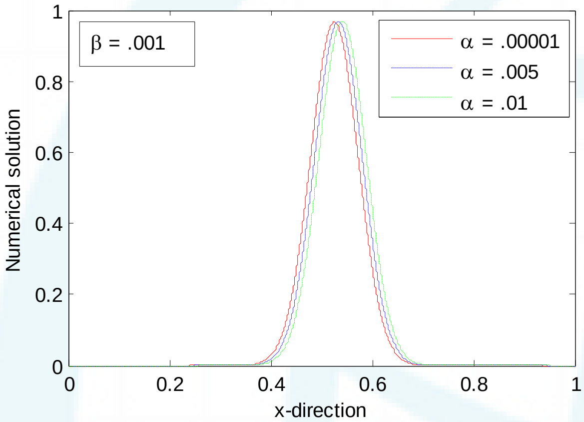

Figure2: The numerical solution for Example 1 at $t = 0.5$.

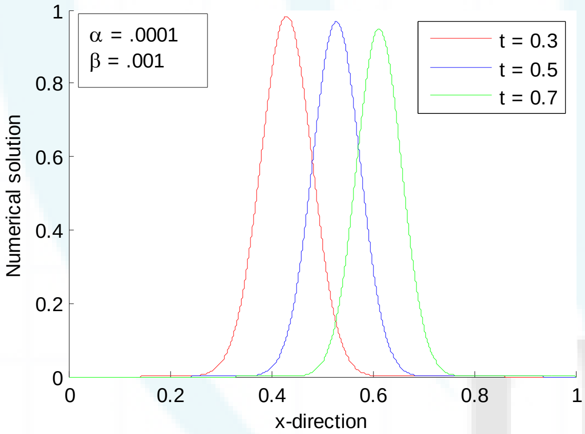

Figure3: The numerical solution of Example 2 for different values of *t*.

Table 3: The maximum absolute error for example 3

$\Delta t \downarrow \Delta x \rightarrow$	1/100	1/200	1/400	1/800
Δx /2	0.0580	0.0289	0.0146	0.0069
Δx /4	0.0291	0.0143	0.0068	0.0033
Δx /8	0.0145	0.0067	0.0032	0.0015
Δx /16	0.0068	0.0033	0.0016	0.0007

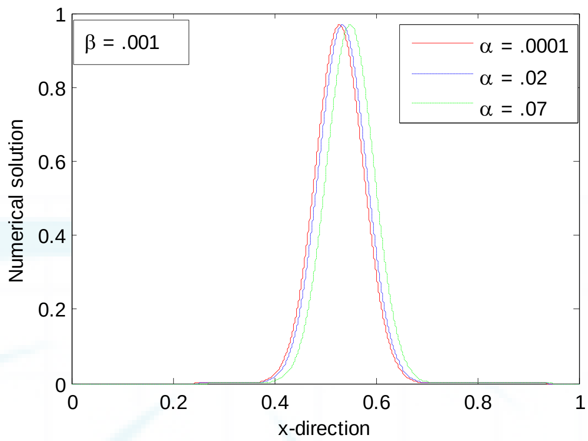

Figure4: The numerical solution for Example 3 at $t = 0.5$.

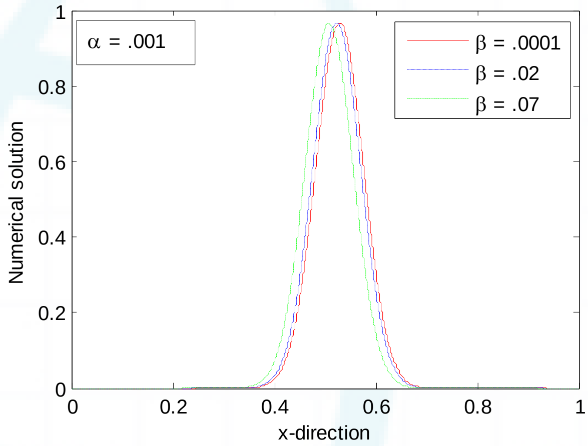

Figure5: The numerical solution for Example 3 at $t = 0.5$.

5. Conclusion

In this paper, a first-order hyperbolic partial differential difference equation for the distribution of neuronal firing based on the Stein's Model [8]. For finding the numerical solution of the initial and boundary value problem, a numerical scheme based on upwind finite difference is developed. The maximum absolute error are computed and tabulated in tables 1-3 for the considered examples with $\alpha = 0.001$ and $\beta = 0.001$. The error table illustrates that the method is first order convergent in temporal and second order spatial directions. Basically in this paper, we compare the results as already discussed by Sharma and Singh [7]. Our results are better due to second order convergence of scheme. The graphs of the solution of the considered examples for different values of point-wise delay and advance are plotted in Figures 1-5 to examine the effect of point-wise delay as well as advance on the solution behavior of the problem. We observe that if we fix α and increase the value of β, impulse moves towards left see (fig.1 and 5) while fixing β and increase the value of α, impulse moves towards right see (fig.2 and 4). Now fixing both α and β, the impulse moves towards right with the time see (fig.3).

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Author Profile

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