The vertex 2i is incident with the edge (2i – 1, 2i) and (2i, 2i + 1). But the edge (2i – 1, 2i) $\notin$ F and the edge (2i, 2i + 1) = { e_i } $\notin$ F – { e_i }.

Similarly (2i + 1) is incident with the edge (2i, 2i + 1) and (2i + 1, 2i + 2). But the edge (2i, 2i + 1) = { e_i } $\notin$ F – { e_i } and the edge (2i + 1, 2i + 2) $\notin$ F.

i.e., the set of edges of F – { e_i } does not contain all the vertices of G.

So F – { e_i } is not an edge cover of $G(Z_n, \varphi)$.

$\Rightarrow$ F is a minimum edge cover of G. ■

Theorem 3.4: The minimum edge cover of $G(Z_n, \varphi)$, when n is even is given by {(0, 1) (2, 3),….,(n – 2, n – 1)}.

Proof: Consider $G(Z_n, \varphi)$, where n is even. Let E denote the edge set of $G(Z_n, \varphi)$

Consider the set F of ordered pairs of vertices given by F = {(0, 1), (2, 3),……,(n – 2, n – 1)}.

Since (2i + 1) – (2i) = 1 $\in$ S, each ordered pair (2i, 2i + 1), $0 \le i \le \frac{n-2}{2}$ in F is an edge of G.

So F becomes a subset of E.

Also the edges in F contain all the vertices of $G(Z_n, \varphi)$.

$\Rightarrow$ F is an edge cover of G.

Now we check for the minimality of F.

Let us consider F – { e_i }, where e_i = (2i, 2i + 1) $\in$ F, for any i = 0, 1, 2…….., $\frac{n-1}{2}$.

Consider the vertices 2i and 2i + 1.

The vertex 2i is incident with the edges (2i – 1, 2i) and (2i, 2i + 1). But the edge (2i – 1, 2i) $\notin$ F and the edge (2i, 2i + 1) = { e_i } $\notin$ F – { e_i }.

Similar is the case with the vertex 2i+1.

So the edges of F – { e_i } does not cover all the vertices of G.

i.e., F – { e_i } is not an edge cover of G.

$\Rightarrow$ F is a minimum edge cover of $G(Z_n, \varphi)$. ■

Corollary 3.5: The edge covering number of $G(Z_n, \varphi)$ is $\frac{n}{2}$ when n is even and $\frac{n+1}{2}$ when n is odd.

Proof: If n is even then the n vertices of $G(Z_n, \varphi)$ can be paired into $\frac{n}{2}$ distinct pairs of vertices (2i, 2i + 1), $0 \le i \le \frac{n-2}{2}$, it follows that the cardinality of F = {(0, 1), (2, 3),….,(n – 2, n – 1)} is $\frac{n}{2}$. i.e., $\beta'(G(Z_n, \varphi)) = \frac{n}{2}$.

If n is odd then the n vertices of $G(Z_n, \varphi)$ can be paired into $\frac{n+1}{2}$ distinct pairs of vertices (2i, 2i + 1), $0 \le i \le \frac{n-1}{2}$.

Hence the cardinality of F = {(0, 1), (2, 3),….,(n – 1, 0)} is $\frac{n+1}{2}$.

i.e., $\beta'(G(Z_n, \varphi)) = \frac{n+1}{2}$. ■

Theorem 3.6: The set of edges {(1, 2), (3, 4),….,(n – 2, n – 1)} forms a minimal edge dominating set of $G(Z_n, \varphi)$, when n is odd.

Proof: Consider $G(Z_n, \varphi)$, when n is odd. Let E denote the edge set of $G(Z_n, \varphi)$.

Let F = {(1, 2), (3, 4),…..,(n – 2, n – 1)}.

Since 2i – (2i – 1) = 1$\in$S, each ordered pair (2i –1, 2i), $0 < i \le \frac{n-1}{2}$ in F is an edge of $G(Z_n, \varphi)$.

$\Rightarrow$ F is a subset of E.

Let (s, t) $\in$ E – F, where s $\ge$ 0 and t $\ne$ s + 1.

Consider the edge (s, s + 1) in F, where s $\ne$ 2i, i = 1, 2…….. $\frac{n-1}{2}$.

Obviously this edge is adjacent with (s, t).

So every edge of E – F is adjacent with at least one edge of F.

$\Rightarrow$ F is an edge dominating set.

We now check for the minimality of F.

Delete an edge e = (1, 2) from F. Then there is an edge (0, 1) $\in$ E, such that it is not adjacent with any edge of F – { e }, because the edge (0, 1) is adjacent with the edges (1, 2) and (0, n – 1) for n > 1 as 1 $\in$ S. Also the edge (0, 1) is adjacent with the edges (0, q) and (1, r),

where 1 < q < (n – 1), 2 < r < n. Let $|q - 0| = k_1$ and $|r - 1| = k_2$, where $k_1, k_2 > 1 \in$ S.

But none of these edges belong to F – { e }, as the edges in F are of the form

(2i – 1, 2i) where i = 1, 2,…., $\frac{n-1}{2}$.

i.e., F – { e } is not an edge dominating set of $G(Z_n, \varphi)$.

Hence F is a minimum edge dominating set of $G(Z_n, \varphi)$. ■

Corollary 3.7: The edge domination number $\gamma'(G(Z_n, \varphi)) = \frac{n-1}{2}$, when n is odd.

Proof: By Theorem 3.6, the minimum edge dominating set of $G(Z_n,\varphi)$ is
F = {(1, 2), (3, 4), ……,(n – 2, n – 1)}.

Since the (n – 1) vertices can be paired into $\frac{n-1}{2}$ distinct pairs of vertices (2i – 1, 2i),

$0 < i \leq \frac{n-1}{2}$, it follows that the cardinality of F is $\frac{n-1}{2}$.

Therefore $\gamma'(G(Z_n,\varphi)) = \frac{n-1}{2}$. ■

Theorem 3.8: The set of edges {(0, 1), (2, 3),……,(n − 2, n − 1)} forms a minimal edge dominating set of $G(Z_n,\varphi)$, when n is even.

Proof: Consider $G(Z_n,\varphi)$, where n is even. Let E denote the edge set of $G(Z_n,\varphi)$.

Let F = {(0, 1), (2, 3),…,(n−2, n – 1)}.

Since (2i +1) – 2i = 1 ∈ S, each ordered pair (2i, 2i + 1), $0 \leq i \leq \frac{n-2}{2}$ in F is an edge of $G(Z_n,\varphi)$. i.e., F is a subset of E in $G(Z_n,\varphi)$.

Let (s, t) ∈ E – F , where s ≥ 0 and t ≠ s + 1.

Consider the edge (s, s + 1) in F, where s ≠ 2i, i = 0, 1, 2,…, $\frac{n-1}{2}$.

Obviously this edge is adjacent with (s, t).

Hence it follows that F is an edge dominating set of $G(Z_n,\varphi)$.

Now we prove that F is minimal. Delete an edge e = (0, 1) from F.

Then the edge (0, 1) belongs to E – F, which is adjacent with (1, 2) and (0, n − 1) for n > 1.

Also it is adjacent with the edges (0, q) and (1, r) where 1 < q < n − 1, 2 < r < n.

Let $|0 - q| = k_1$ and $|1 - r| = k_2$ where $k_1, k_2 > 1 \in S$.

But none of these edges belong to F – { e } as the edges in F are of the form

(2i, 2i + 1) where i = 1, 2,…., $\frac{n-2}{2}$.

i.e., F – { e } is not an edge dominating set.

Hence F is a minimum edge dominating set of $G(Z_n,\varphi)$.

Corollary 3.9: The edge domination number $\gamma'(G(Z_n,\varphi)) = \frac{n}{2}$, when n is even.

Proof: By Theorem 3.8, a minimum edge dominating set of $G(Z_n,\varphi)$ when n is even is given by F = {(0, 1), (2, 3),……….,(n – 2, n – 1)}.

The n vertices {0, 1, 2, 3,….,n – 1} can be paired into $\frac{n}{2}$ distinct pairs of vertices (2i, 2i + 1), $0 \leq i \leq \frac{n-2}{2}$.

Hence it follows that the cardinality of F is $\frac{n}{2}$.

i.e., $\gamma'(G(Z_n,\varphi)) = \frac{n}{2}$. ■

4. Future Scope

There are several other dominating parameters which can be applied on Euler Totient Cayley Graph. The theorems which are derived in this paper can be applied practically for solving graph theoritical problems.

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