

Theorem 3.22. Let X be a regular space. Then for a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- (1) f is weakly b - δ -open;
- (2) For each δ -open set A in X , $f(A)$ is b - δ -open in Y ;
- (3) For any set B of Y and any δ -closed set A in X containing $f^{-1}(B)$, there exists a b - δ -closed set F in Y containing B such that $f^{-1}(F) \subseteq A$.

Proof. (1) \Rightarrow (2): Let A be a δ -open set in X . Since X is regular, by Theorem 3.10, f is b - δ -open and A is open. Therefore $f(A)$ is b - δ -open in Y .

(2) \Rightarrow (3): Let B be any set in Y and A a δ -closed set in X such that $f^{-1}(B) \subseteq A$. Since $X-A$ is δ -open in X , by (2), $f(X-A)$ is b - δ -open in Y . Let $F=Y-f(X-A)$. Then F is b - δ -closed and $B \subseteq F$.

Now $f^{-1}(F) = f^{-1}(Y-f(X-A)) = X-f^{-1}(f(X-A)) \subseteq A$.

(3) \Rightarrow (1): Let B be any set in Y . Let $A = \delta\text{-cl}(f^{-1}(B))$. Since X is regular, A is a δ -closed set in X and $f^{-1}(B) \subseteq A$. Then there exists a b - δ -closed set F in Y containing B such that $f^{-1}(F) \subseteq A$. Since F is b - δ -closed, $f^{-1}(b\text{-}\delta\text{-cl}(B)) \subseteq f^{-1}(F) \subseteq A = \delta\text{-cl}(f^{-1}(B))$. Therefore by Theorem 3.5, f is weakly b - δ -open.

4. Weakly b - δ Closed Functions

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be b - δ -closed if for each closed set F of (X, τ) , $f(F)$ is b - δ -closed.

Definition 4.2. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be weakly b - δ -closed if $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$ for each closed set F of (X, τ) .

Theorem 4.3: Every b - δ -closed function is also weakly b - δ -closed function.

Proof: Follows from Definitions.

The converse of above theorem need not be true as shown in the following example.

Example 4.4. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then $B\delta C(X) = \{\emptyset, \{b\}, \{a, c\}, X\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then f is a weakly b - δ -closed function which is not b - δ -closed, since for $U = \{b, c\}$ and $\text{int}(U) = \{c\}$, $f(U)$ is not b - δ -closed in (X, τ) .

Theorem 4.5. Every b - δ -closed function is also b - θ -closed function.

Proof. Every b - δ -closed is also b - θ -closed set. Hence the Proof follows from the Definitions of b - δ -closed and b - θ -closed sets.

Theorem 4.6. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- (1) f is weakly b - δ -closed;
- (2) $b\text{-}\delta\text{-cl}(f(U)) \subseteq f(\text{cl}(U))$ for each open set U in (X, τ) .

Proof. (1) \Rightarrow (2): Let U be an open set in X . Since $\text{cl}(U)$ is a closed set and $U \subseteq \text{int}(\text{cl}(U))$, we have $b\text{-}\delta\text{-cl}(f(U)) \subseteq b\text{-}\delta\text{-cl}(f(\text{int}(\text{cl}(U)))) \subseteq f(\text{cl}(U))$.

(2) \Rightarrow (1): Let F be a closed set of X . Then, we have $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(\text{cl}(\text{int}(F))) \subseteq f(\text{cl}(F)) = f(F)$ and hence f is weakly b - δ -closed.

Corollary 4.7. A bijective function $f : (X, \tau) \rightarrow (Y, \sigma)$, is weakly b - δ -open if and only if f is weakly b - δ -closed.

Proof. This is an immediate consequence of Theorems 3.6 and 4.6.

Theorem 4.8. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- (1) f is weakly b - δ -closed,
- (2) $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$ for each preclosed set F in (X, τ) ,
- (3) $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$ for each α -closed set F in (X, τ) ,
- (4) $b\text{-}\delta\text{-cl}(f(\text{int}(\text{cl}(U)))) \subseteq f(\text{cl}(U))$ for each subset U in (X, τ) ,
- (5) $b\text{-}\delta\text{-cl}(f(U)) \subseteq f(\text{cl}(U))$ for each preopen set U in (X, τ) .

Proof: Follows from Definitions

Theorem 4.9. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- 1) f is weakly b - δ -closed,
- 2) $b\text{-}\delta\text{-cl}(f(U)) \subseteq f(\text{cl}(U))$ for each regular open set U in (X, τ) ,
- 3) For each subset F in Y and each open set U in X with $f^{-1}(F) \subseteq U$, there exists a b - δ -open set A in Y with $F \subseteq A$ and $f^{-1}(A) \subseteq \text{cl}(U)$,
- 4) For each point y in Y and each open set U in X with $f^{-1}(y) \subseteq U$, there exists a b - δ -open set A in Y containing y and $f^{-1}(A) \subseteq \text{cl}(U)$.

Proof. (1) \Rightarrow (2): Let U be a regular open subset of (X, τ) . Then U is open and so $U = \text{int}(U)$. Since $\text{cl}(U)$ is closed and f is weakly b - δ -closed, $b\text{-}\delta\text{-cl}(f(U)) = b\text{-}\delta\text{-cl}(f(\text{int}(U))) \subseteq b\text{-}\delta\text{-cl}(f(\text{int}(\text{cl}(U)))) \subseteq f(\text{cl}(U))$. Hence $b\text{-}\delta\text{-cl}(f(U)) = f(\text{cl}(U))$.

(2) \Rightarrow (3): Let F be a subset of Y and U an open set in X with $f^{-1}(F) \subseteq U$. Then $f^{-1}(F) \cap \text{cl}(X - \text{cl}(U)) = \emptyset$ and consequently, $F \cap f(\text{cl}(X - \text{cl}(U))) = \emptyset$. Since $X - \text{cl}(U)$ is regular open, $F \cap b\text{-}\delta\text{-cl}(f(X - \text{cl}(U))) = \emptyset$. Let $A = Y - b\text{-}\delta\text{-cl}(f(X - \text{cl}(U)))$. Then A is a b - δ -open set with $F \subseteq A$ and we

have $f^{-1}(A) \subseteq X - f^{-1}(b-\delta\text{-cl}(f(Y - c\text{ l}(U)))) \subseteq X - f^{-1}(Y - c\text{ l}(U)) \subseteq c\text{ l}(U)$.

(3) \Rightarrow (4): This is obvious.

(4) \Rightarrow (1): Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \subseteq X - F$, by (4) there exists a $b-\delta$ -open set A in Y with $y \in A$ and $f^{-1}(A) \subseteq c\text{ l}(X - F) = X - \text{int}(F)$. Therefore $A \cap f(\text{int}(F)) = \emptyset$, so that $y \notin b-\delta\text{-cl}(f(\text{int}(F)))$. Thus $b-\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$. Hence f is weakly $b-\delta$ -closed.

Theorem 4.10. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a bijective weakly $b-\delta$ -closed function, then for every subset F in Y and every open set U in X with $f^{-1}(F) \subseteq U$, there exists a $b-\delta$ -closed set B in Y such that $F \subseteq B$ and $f^{-1}(B) \subseteq \text{cl}(U)$.

Proof. Let F be a subset of Y and U be an open subset of X with $f^{-1}(F) \subseteq U$. Put $B = b-\delta\text{-cl}(f(\text{int}(c\text{ l}(U))))$. Then B is a $b-\delta$ -closed set in (Y, σ) such that $F \subseteq B$, since $F \subseteq f(U) \subseteq f(\text{int}(c\text{ l}(U))) \subseteq b-\delta\text{-cl}(f(\text{int}(c\text{ l}(U)))) = B$. Since f is weakly $b-\delta$ -closed, by Theorem 4.6, we have $f^{-1}(B) \subseteq \text{cl}(U)$.

Theorem 4.11. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a weakly $b-\delta$ -closed surjection and all pairs of disjoint fibers are strongly separated then (Y, σ) is $b-T_2$.

Proof. Let y and z be two points in Y . Let U and V be open set in (X, τ) such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$ with $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since f is weakly $b-\delta$ -closed, by Theorem 4.9, there are $b-\delta$ -open sets F and B in (Y, σ) such that $y \in F$ and $z \in B$, $f^{-1}(F) \subseteq \text{cl}(U)$ and $f^{-1}(B) \subseteq \text{cl}(V)$.

Therefore $F \cap B = \emptyset$, because $\text{cl}(U) \cap \text{cl}(V) = \emptyset$ and f is surjective. Since every $b-\delta$ -open is b -open. Then (Y, σ) is $b-T_2$.

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