

Theorem 3.22. Let X be a regular space. Then for a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- (1) f is weakly b - δ -open;
- (2) For each δ -open set A in X , $f(A)$ is b - δ -open in Y ;
- (3) For any set B of Y and any δ -closed set A in X containing $f^{-1}(B)$, there exists a b - δ -closed set F in Y containing B such that $f^{-1}(F) \subseteq A$.

Proof. (1) \Rightarrow (2): Let A be a δ -open set in X . Since X is regular, by Theorem 3.10, f is b - δ -open and A is open. Therefore $f(A)$ is b - δ -open in Y .

(2) \Rightarrow (3): Let B be any set in Y and A a δ -closed set in X such that $f^{-1}(B) \subseteq A$. Since $X-A$ is δ -open in X , by (2), $f(X-A)$ is b - δ -open in Y . Let $F=Y-f(X-A)$. Then F is b - δ -closed and $B \subseteq F$.

Now $f^{-1}(F) = f^{-1}(Y-f(X-A)) = X-f^{-1}(f(X-A)) \subseteq A$.

(3) \Rightarrow (1): Let B be any set in Y . Let $A = \delta\text{-cl}(f^{-1}(B))$. Since X is regular, A is a δ -closed set in X and $f^{-1}(B) \subseteq A$. Then there exists a b - δ -closed set F in Y containing B such that $f^{-1}(F) \subseteq A$. Since F is b - δ -closed, $f^{-1}(b\text{-}\delta\text{-cl}(B)) \subseteq f^{-1}(F) \subseteq A = \delta\text{-cl}(f^{-1}(B))$. Therefore by Theorem 3.5, f is weakly b - δ -open.

4. Weakly b - δ Closed Functions

Definition 4.1. A functions $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be b - δ -closed if for each closed set F of (X, τ) , $f(F)$ is b - δ -closed.

Definition 4.2. A functions $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be weakly b - δ -closed if $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$ for each closed set F of (X, τ) .

Theorem 4.3: Every b - δ -closed function is also weakly b - δ -closed function.

Proof: Follows from Definitions.

The converse of above theorem need not be true as shown in the following example.

Example 4.4. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then $B\delta C(X) = \{\emptyset, \{b\}, \{a, c\}, X\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then f is a weakly b - δ -closed function which is not b - δ -closed, since for $U = \{b, c\}$ and $\text{int}(U) = \{c\}$, $f(U)$ is not b - δ -closed in (X, τ) .

Theorem 4.5. Every b - δ -closed function is also b - θ -closed function.

Proof. Every b - δ -closed is also b - θ -closed set. Hence the Proof follows from the Definitions of b - δ -closed and b - θ -closed sets.

Theorem 4.6. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- (1) f is weakly b - δ -closed;
- (2) $b\text{-}\delta\text{-cl}(f(U)) \subseteq f(\text{cl}(U))$ for each open set U in (X, τ) .

Proof. (1) \Rightarrow (2): Let U be an open set in X . Since $\text{cl}(U)$ is a closed set and $U \subseteq \text{int}(\text{cl}(U))$, we have $b\text{-}\delta\text{-cl}(f(U)) \subseteq b\text{-}\delta\text{-cl}(f(\text{int}(\text{cl}(U)))) \subseteq f(\text{cl}(U))$.

(2) \Rightarrow (1): Let F be a closed set of X . Then, we have $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(\text{cl}(\text{int}(F))) \subseteq f(\text{cl}(F)) = f(F)$ and hence f is weakly b - δ -closed.

Corollary 4.7. A bijective function $f : (X, \tau) \rightarrow (Y, \sigma)$, is weakly b - δ -open if and only if f is weakly b - δ -closed.

Proof. This is an immediate consequence of Theorems 3.6 and 4.6.

Theorem 4.8. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- (1) f is weakly b - δ -closed,
- (2) $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$ for each preclosed set F in (X, τ) ,
- (3) $b\text{-}\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$ for each α -closed set F in (X, τ) ,
- (4) $b\text{-}\delta\text{-cl}(f(\text{int}(\text{cl}(U)))) \subseteq f(\text{cl}(U))$ for each subset U in (X, τ) ,
- (5) $b\text{-}\delta\text{-cl}(f(U)) \subseteq f(\text{cl}(U))$ for each preopen set U in (X, τ) .

Proof: Follows from Definitions

Theorem 4.9. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent:

- 1) f is weakly b - δ -closed,
- 2) $b\text{-}\delta\text{-cl}(f(U)) \subseteq f(\text{cl}(U))$ for each regular open set U in (X, τ) ,
- 3) For each subset F in Y and each open set U in X with $f^{-1}(F) \subseteq U$, there exists a b - δ -open set A in Y with $F \subseteq A$ and $f^{-1}(A) \subseteq \text{cl}(U)$,
- 4) For each point y in Y and each open set U in X with $f^{-1}(y) \subseteq U$, there exists a b - δ -open set A in Y containing y and $f^{-1}(A) \subseteq \text{cl}(U)$.

Proof. (1) \Rightarrow (2): Let U be a regular open subset of (X, τ) . Then U is open and so $U = \text{int}(U)$. Since $\text{cl}(U)$ is closed and f is weakly b - δ -closed, $b\text{-}\delta\text{-cl}(f(U)) = b\text{-}\delta\text{-cl}(f(\text{int}(U))) \subseteq b\text{-}\delta\text{-cl}(f(\text{int}(\text{cl}(U)))) \subseteq f(\text{cl}(U))$. Hence $b\text{-}\delta\text{-cl}(f(U)) = f(\text{cl}(U))$

(2) \Rightarrow (3): Let F be a subset of Y and U an open set in X with $f^{-1}(F) \subseteq U$. Then $f^{-1}(F) \cap \text{cl}(X - \text{cl}(U)) = \emptyset$ and consequently, $F \cap f(\text{cl}(X - \text{cl}(U))) = \emptyset$. Since $X - \text{cl}(U)$ is regular open, $F \cap b\text{-}\delta\text{-cl}(f(X - \text{cl}(U))) = \emptyset$. Let $A = Y - b\text{-}\delta\text{-cl}(f(X - \text{cl}(U)))$. Then A is a b - δ -open set with $F \subseteq A$ and we

have $f^{-1}(A) \subseteq X - f^{-1}(b-\delta\text{-cl}(f(Y - c\text{ l}(U)))) \subseteq X - f^{-1}(Y - c\text{ l}(U)) \subseteq c\text{ l}(U)$.

(3) \Rightarrow (4): This is obvious.

(4) \Rightarrow (1): Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \subseteq X - F$, by (4) there exists a $b-\delta$ -open set A in Y with $y \in A$ and $f^{-1}(A) \subseteq c\text{ l}(X - F) = X - \text{int}(F)$. Therefore $A \cap f(\text{int}(F)) = \emptyset$, so that $y \notin b-\delta\text{-cl}(f(\text{int}(F)))$. Thus $b-\delta\text{-cl}(f(\text{int}(F))) \subseteq f(F)$. Hence f is weakly $b-\delta$ -closed.

Theorem 4.10. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a bijective weakly $b-\delta$ -closed function, then for every subset F in Y and every open set U in X with $f^{-1}(F) \subseteq U$, there exists a $b-\delta$ -closed set B in Y such that $F \subseteq B$ and $f^{-1}(B) \subseteq \text{cl}(U)$.

Proof. Let F be a subset of Y and U be an open subset of X with $f^{-1}(F) \subseteq U$. Put $B = b-\delta\text{-cl}(f(\text{int}(c\text{ l}(U))))$. Then B is a $b-\delta$ -closed set in (Y, σ) such that $F \subseteq B$, since $F \subseteq f(U) \subseteq f(\text{int}(c\text{ l}(U))) \subseteq b-\delta\text{-cl}(f(\text{int}(c\text{ l}(U)))) = B$. Since f is weakly $b-\delta$ -closed, by Theorem 4.6, we have $f^{-1}(B) \subseteq \text{cl}(U)$.

Theorem 4.11. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a weakly $b-\delta$ -closed surjection and all pairs of disjoint fibers are strongly separated then (Y, σ) is $b-T_2$.

Proof. Let y and z be two points in Y . Let U and V be open set in (X, τ) such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$ with $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since f is weakly $b-\delta$ -closed, by Theorem 4.9, there are $b-\delta$ -open sets F and B in (Y, σ) such that $y \in F$ and $z \in B$, $f^{-1}(F) \subseteq \text{cl}(U)$ and $f^{-1}(B) \subseteq \text{cl}(V)$.

Therefore $F \cap B = \emptyset$, because $\text{cl}(U) \cap \text{cl}(V) = \emptyset$ and f is surjective. Since every $b-\delta$ -open is b -open. Then (Y, σ) is $b-T_2$.

References

- [1] Abd El-Monsef M.E., El-Deeb S. N., Mahmoud R. A., "β-open sets and β-continuous mappings," Bull. Fac. Sci. Assiut Univ. (12), pp. 77-90, 1983.
- [2] Andrijević D., "Semi-preopen sets," Mat. Vesnik 38 (1), pp.24-32, 1986.
- [3] Andrijević D., "On b-open sets," Mat. Vesnik (48), pp. 59-64, 1996.
- [4] Levine N., "Strong continuity in topological spaces," Amer. Math. Monthly 67 (3), pp.269, 1960.
- [5] Levine N., "Semi-open sets and semi-continuity in topological spaces," Amer. Math. Monthly (70), pp. 36-41, 1963.
- [6] Mashhour A. S., Abd El-Monsef M. E., El-Deeb S. N., "On precontinuous and weak precontinuous functions," Proc. Math. Phys. Soc. Egypt (53), pp.47-53, 1982.
- [7] Njåstad O., On some classes of nearly open sets, Pacific J. Math. (15), 961-970, 1965.
- [8] Padmanaban S., "On b-δ-open sets in topological spaces," Int. Jr. of Mathematical Sciences & Applications (3), pp.365-371, 2013.

- [9] Singal M.K., Singal A. R., "Almost continuous mappings," Yokohama Math. J. (16), 63-73, 1968
- [10] Steen L. A., Seebach J.A, Jr., Counterexamples in Topology, Holt, Reinhart and Winston, Inc., New York, 1970.
- [11] Veličko N.V., "H-closed topological spaces," Amer. Math. Soc. Transl. Ser. 2 (78), pp.103-118, 1968.

Author Profile



Dr. S. Anuradha is working as Prof & Head, PG & Research Dept of Maths, Hindusthan Arts College. She has more than 20 years academic teaching experience and 15 years research experience. She published 25 research articles in reputed international journals and also have attended 62 Seminars/Workshop/Conference and presented the papers in the same. She guided 13 M.Phil students and guiding 6 Ph.D students.



S. Padmanaban is working as Assistant Professor in Department of Science & Humanities, Karpagam Institute of Technology. He is having 19 years of teaching experience in engineering colleges and published 6 papers in reputed international journals.