Common Fixed Point of Six Self Maps in Fuzzy Metric Space

Dharamvir Singh Vashisth[1], Raghu Nandan Patel[2], Manoj Kumar Tiwari[3]

[1]Department of Mathematics, Kalinga University, Raipur, Chhattisgarh, India

[2]Department of Mathematics, Government Mukut Dhar Pandey College, Katghora, Chhattisgarh, India

[3]Government Girls Polytechnic, Bilaspur, Chhattisgarh, India

Abstract: ***In this paper, we introduce the concept of semi weakly compatibility of maps in fuzzy metric space and apply it to prove the results on existence of unique common fixed point of six self maps satisfying an implicit relation. The implicit relation used in this paper is similar but not the same as employed by Popa [8] in 2002. All the results of this paper are new/generalizations of the earlier results.***

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1. Introduction

The concept of fuzzy sets was first given by Zadeh in 1965. Then Kramosil and Michalek[8] introduced the concept of fuzzy metric space and George and Veeramani[4] modified the notion of fuzzy metric with help of continuous t-norms. The improving commutativity in fixed point theorems by using weakly commuting maps in metric space was initiated by Sessa [10] .Later on, this method was enlarged to compatible maps by Jungck[5] . Cho[2,3] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. Singhet.al.[12] proved fixed point theorems in a fuzzy metric space. Recently in 2012 Jain et.al.[6] proved various fixed point theorems using the concept of semi compatible mapping This concept is most general concept among all commutativity concepts. In this scenario every pair of commuting self maps is compatible, every pair of compatible self-maps is weakly compatible but the converse is not true always. Similarly, every semi compatible pair of self-maps is weakly compatible but the converse is not true always.

The main objective of this paper is to introduce a new class of commutativity of maps namely, semi weakly compatibility of maps in fuzzy metric space. Also, using this concept along with the concepts of weakly compatibility and semi-compatibility of maps satisfying an implicit relation, we have obtained some fixed point theorems in the setting of fuzzy metric space. In the sequel, a characterization of such implicit relation is also derived in the linear form and used the same to establish some results regarding fixed point in fuzzy metric space. The idea of fuzzy 2- metric space and fuzzy 3- metric space were used by Sushil Sharma [11] and obtained some fruitful results.

Definition 1.1 Let X be any set . A Fuzzy set A in X is a function with domain X and Values in [0,1].

Definition 1.2 A Binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norms if an topological monoid with unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$, for all a, b, c, d in [0,1].

Definition 1.3 The triplet (X,M, *) is said to be a Fuzzy metric space if , X is an arbitrary set , * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions; for all x,y,z in X and s,t > 0,

(i) $M(x,y,0) = 0$, $M(x,y,t)>0$,
(ii) $M(x,y,t) = 1$,for all $t > 0$ if and only if x=y,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t) * M(y,z,s) \le M(x,z, t+s)$,
(v) $M(x,y,t) : [0,\infty) \to [0,1]$ is left continuous.

Definition 1.4 A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is called a Cauchy Sequence if , $\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$ for every t.>0 and for each p>0. A fuzzy metric space(X, M,*) is Complete if ,every Cauchy sequence in x converges in X.

Definition 1.5 A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is said to be Convergent to x in X if , $\lim_{n\to\infty} M(x_n, x, t) = 1$, for each t>0.

Definition 1.6 Two self mappings P and Q of a fuzzy metric space (X,M,*) are said to be Compatible , if $\lim_{n\to\infty} M(PQx_n,QPx_n,t) =1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$, for some z in X .

Defnition 1.7. Two self mappings A and S on a fuzzy metric space (X;M; *) are said to be weakly compatible if they commute at their coincidence points, that is if for $x \in X$, $Ax = Sx$ implies that $M(ASx; SAx; t) = 1$ for all $t > 0$

*** Two self-mappings A and S on a fuzzy metric space (X;M; _) are compatible, then they are weakly compatible, but the converse is not true.**

Defnition 1.8. Two self mappings A and S on a fuzzy metric space (X;M; *) are said to be semi-compatible if $\lim_{n\to 1} M(ASx_n; Sx; t) = 1$; $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to 1} Ax_n = \lim_{n\to 1} Sx_n = x \in X$.

**** It follows that if (A, S) is semi-compatible and Ay = Sy, then ASy = SAy (on taking $x_n = y$ for all n). Thus if the pair (A, S) is semi-compatible, then it is weakly compatible, but the converse is not true always.**

Lemma 1.9 Let $\{y_n\}$ is a sequence in an FM- space . If there exists a positive number k<1 such that $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n,t)$, t>0 , n $\in$ N, then $\{y_n\}$ is a Cauchy sequence in X .

Lemma 1.10 If for two points x, y in X and a positive number k < 1 $M(x,y,kt) \geq M(x,y,t)$, then x = y. Next we give some properties of compatible mappings of type (A-1) which will be used in our main theorem.

Remark 1.11 A class of implicit relation. Let ϕ be the set of all real continuous functions ϕ: $(R_+)^4 \to R$, non-decreasing in first argument and satisfying the following conditions:

(i) For u; v ≥ 0 , ϕ (u; v; v; u) ≥ 0 or ϕ (u; v; u; v) ≥ 0 implies that u ≥ v.

(ii) ϕ (u; u; 1; 1) ≥ 0 implies that u ≥1.

2. Main Result

We prove the following theorem.

Theorem 2.1 : Let A, B, S, T, P and Q be self maps on a complete fuzzy metric space (X, M, *) where * is continuous t norm defined by a*b = min {a,b} satisfying the following conditions

(2.1.1). $A(X) \subseteq QT(X)$ and $B(X) \subseteq PS(X)$

(2.1.2). one of A or PS is continuous;

(2.1.3). For each x,y $\in$X and t > 0 ,

$M(Ax ,By,t) \geq \phi[\min\{M(PSx, QTy,t) , M(By, PSx,t) , M(Ax, PSx,t), M(By, QTy ,t)\}]$

Where ϕ : [0,1]→[0,1] is a continuous function such that $\phi(1) = 1$, $\phi(0) =0$ and $\phi(a) > a$, for each 0<a<1 .

(2.1.4) the pairs (P, S) and (Q, T) are commuting mappings.

(2.1.5) the pairs (P, A),(S, A),(Q, B) and (T, B) are semi weakly compatible mappings.

If the pair (A, PS) is semi-compatible and (B, QT) is weakly compatible, then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be any arbitrary point as $A(X) \subseteq QT(X)$ and $B(X) \subseteq PS(X)$, there exist $x_1; x_2 \in X$ such that $Ax_0 = QTx_1$, $Bx_1 = PSx_2$. Inductively, we can construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = QTx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = PSx_{2n+2}$, for n = 0, 1, 2, . . .

Now using (iii) with $x = x_{2n}$; $y = x_{2n+1}$, we get

$M(Ax_{2n} , Bx_{2n+1}, t) \geq\phi [\min \{ M (PSx_{2n}, QTx_{2n+1}, t) , M (Bx_{2n}, PSx_{2n+1},t) , M(Ax_{2n},PSx_{2n},t) , M(B x_{2n+1}, QT x_{2n+1} ,t)\}]$

$M(y_{2n+1},y_{2n+2} , t) \geq\phi [\min \{M (y_{2n}, y_{2n+1},t) , M(y_{2n+1} , y_{2n+1},t) , M(y_{2n+1},y_{2n},t), M(y_{2n+2}, y_{2n+1},t)\}]$

$M(y_{2n+1},y_{2n+2} , t) \geq\phi [\min \{ M (y_{2n}, y_{2n+1},t) , 1, M(y_{2n+1} , y_{2n},t), M(y_{2n+2}, y_{2n+1} ,t)\}]$

Hence , by the definition of ϕ , we get $M(y_{2n+1},y_{2n+2} , t) \geq M (y_{2n}, y_{2n+1},t)$

Similarly , we have $M(y_{2n+2} , y_{2n+3} ,t) \geq M(y_{2n+1},y_{2n+2},t)$,

In general $M(y_{n+1},y_n,t) \geq M(y_n, y_{n-1},t)$

Therefore, $\{ M(y_{n+1},y_n,t) \}$ is an increasing sequence of positive real numbers.

Hence by lemma 1.9, $\{y_n\}$ is a Cauchy sequence in X. By the completeness of X, $\{y_n\}$ and its all subsequences $\{Ax_{2n}\}$; $\{Bx_{2n+1}\}$; $\{PSx_{2n}\}$; $\{QTx_{2n+1}\}$ are also, converges to some point say u$\in$ X.

Suppose PS is continuous: then, we have $PSAx_{2n} \to PSu$; $(PS)^2x_{2n} \to PSu$. By semi-compatibility of the pair (A, PS) of maps, we have $\lim_{n\to 1}APSx_{2n} = PSu$.

Using (2.1.3) with $x = PSx_{2n}$; $y = x_{2n+1}$, we have

$M(APSx_{2n} ,Bx_{2n+1},t) \geq \phi[\min\{M(PSPSx_{2n}, QTx_{2n+1},t) , M(Bx_{2n+1}, PSPSx_{2n},t), M(APSx_{2n}, PSPSx_{2n},t), M(Bx_{2n+1}, QTx_{2n+1} ,t)\}]$

Letting n→∞, we have

$M(PSu, u,t) \geq \phi[\min\{M(PSu, u,t) ,M(u, PSu,t), M(Au, u,t), M(u, u ,t)]$

$M(PSu, u,t) \geq \phi[\min\{M(PSu, u,t) . M(u, PSu,t)\}, 1,1]$

Using 1.11 (ii),we get M(PSu; u; t) ≥ 1, for all t > 0, which gives PSu = u.

Again by putting x = u; $y = x_{2n+1}$ in (2.1.3), we obtain

$M(Au ,Bx_{2n+1},t) \geq \phi[\min\{M(PSu, QTx_{2n+1},t) , M(Bx_{2n+1}, PSu,t), M(Au, PSu,t), M(Bx_{2n+1}, Ax_{2n+1} ,t)\}]$

$M(Au ,u,t) \geq \phi[\min\{M(PSu, u,t) , M(u, PSu,t) ,M(Au,PSu,t), M(u, u,t)\}]$

$M(Au ,u,t) \geq \phi[\min\{M(u, u,t) , M(u, u,t) ,M(Au,u,t), M(u, u,t)\}]$

Taking limit as n→∞, and using 1.11(i),we get M(Au; u; t) ≥ 1, for all t > 0,

Hence Au = u = PSu .

Since $A(X) \subseteq QT(X)$, there exists w $\in$ X such that Au = PSu = u = QTw.

By putting $x = x_{2n}$; y = w in (2.1.3), we obtain

$M(Ax_{2n} ,Bw,t) \geq\phi[\min\{M(PSx_{2n}, QTw,t) , M(Bw, PSx_{2n},t) , M(Ax_{2n}, PSx_{2n},t), M(Bw, QTw ,t)]$

$M(u ,Bw,t) \geq\phi[\min\{M(u, QTw,t) , M(Bw, u,t) , M(u, u,t), M(Bw, QTw ,t)]$

$M(u ,Bw,t) \geq\phi[\min\{M(u, u,t) , M(Bw, u,t) , M(u, u,t), M(Bw, u ,t)]$

Taking limit as n→∞, and using 1.11(i), we get u = Bw. Therefore Bw = QTw = u. Since the pair (B, QT) is weakly compatible mappings, we get QTBw = BQTw, that is Bu = QTu.

Now by putting x = y = u in (2.1.3), and using 1.11(ii), we have Bu = Au.

Therefore u = Au = PSu = Bu = QTu, that is, u is a common fixed point of the maps A, B, PS and QT.

Similarly it can be proved that if the map A is continuous then u is the common fixed point of the mas A, B, PS and QT.

3. Uniqueness

Let z be another common fixed point of the maps A, B, PS and QT.

Putting x = u and y = z in (2.1.3) and using 2.1(i), we get

$M(Au ,Bz,t) \geq \phi[\min\{M(PSu, QTz,t) , M(Bz, PSu,t) , M(Au, PSu,t), M(Bz, QTz ,t)\}]$

$M(u, z, t) \geq \phi[\min\{M(u, z, t), M(z, u, t), M(u, u, t), M(z, z, t)\}]$

that is $u = z$.

Therefore u is the unique common fixed point of the self maps A, B, PS and QT in fuzzy metric space X.

From (2.1.4 and 2.1.5)), we have $Pz = P(PSz) = P(SPz) = (PS)Pz$; $Pz = PAz = APz$ and $Sz = S(PSz) = (SP)Sz = (PS)Sz$; $Sz = SAz = ASz$, implies that Pz and Sz are common fixed points of the maps PS and A.

Therefore $z = Pz = Sz = Az = PSz$. Similarly, Qz and Tz are common fixed points of the maps QT and B, therefore $z = Qz = Tz = Bz = QTz$.

Hence z is the common fixed point of the maps A, B, S, T, P and Q.

Further since z is the unique common fixed point of the maps A, B, PS and QT, consequently it is the unique common fixed point of the maps A, B, S, T, P and Q.

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