A Steiner Problem in Coxeter Graph Which is NP – Complete

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Abstract: ***Our objects in this work is to obtain complexity theory, a specific problem known as Steiner problem in Coxeter graph which is NP complete and also it is worth mentioning that our result every full component of a Steiner tree contains almost 4 terminals involved therein.***

Keywords: Steiner tree, satisfiability, NP-complete, Coxeter graph

1. Introduction

A Steiner minimum tree in a graph with R-terminals is interior points. The Steiner tree (ST) problem in graph called for brevity ST, defined in decisional form as follows:

- An undirected graph $G=(V,E)$
- A subset of the vertices $R \subseteq V$, called terminal nodes.
- A number $K \in N$.

There is a subtree of G that includes all the vertices of R. (i.e., a spanning tree for R) and that contains at most K edges.
Steiner tree problem has many applications especially when we have to plan a connectivity structure among different terminal points. For example, when we want to find an optimal way to build roads and railways to connect, a set of cities or decide routing policies over the internet for multicast traffic, usually from a source to many destinations.

For many decision problems no polynomial time algorithm is known. Nevertheless some of these problems have a property which is not inherent to every decision problem, there exists algorithm which, if presented with an instance of the problem [i.e., a graph G with terminal set K, and a bound B, respectively a Boolean Formula F]. These algorithms verify in polynomial time whether x a valid solution is. The decision problem with this property forms the NP. In this work, we propose an NP-completeness result for the Steiner problem in coxeter graphs.

2. Preliminaries

Definition 2.1

A Steiner tree is a tree in a distance graph which spans a given subset of vertices (Steiner point with the minimum total distance on its edges).

Figure 1: Steiner Tree

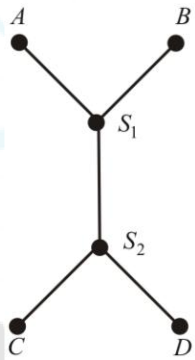

Figure 2: Steiner Tree

Two Steiner points S_1 and S_2.

The Steiner Tree problem

Definition

Let $G=(V,E)$ be an evaluated graph. Let T be a set of terminal nodes that should be connected. The Steiner problem consists of finding a tree of G containing all terminal nodes T with a minimum weight. The optimal tree can contain other nodes called Steiner nodes in the set = $T \setminus V$. We note that two special cases of Steiner problem are solved polynomialy.

If $|T|=2$, then the Steiner problem is equivalent to shortest path.

If $T=V$ then the Steiner problem is equivalent to the minimum spanning tree problem.

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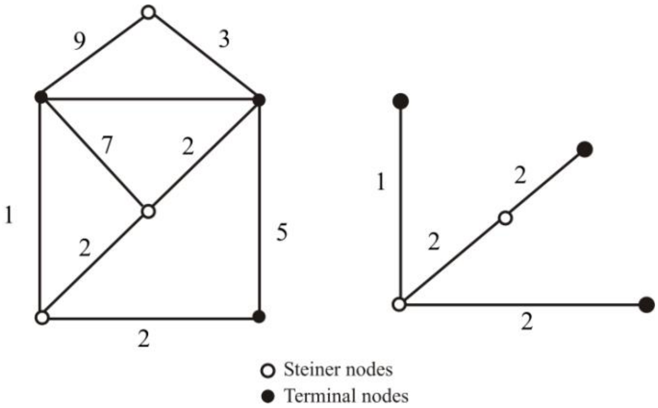

Figure 3: An example of the Steiner tree problem

Definition 2.2

Let a connected graph $G=(V,E)$ and a set $K \subseteq Y$ of terminals, then the **Steiner minimum tree** for K in G that is Steiner tree τ for K such that

$$|E(T)| = \min\{E(T')/T' is\, a\, steiner tree\, for\, K\, in G\}$$

In the Steiner minimum tree problem, the vertices are divided into two parts, terminals and non-terminal vertices. The terminals are the given vertices which must be included in the solution.

Example 2.2

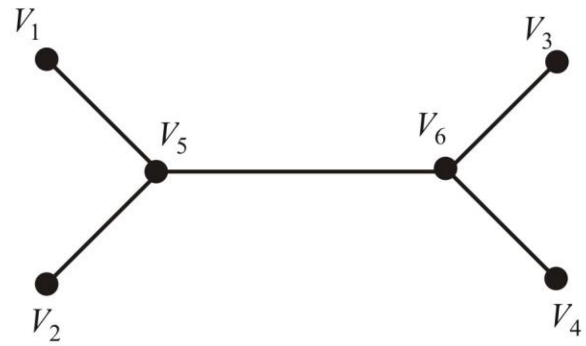

Figure 4: Steiner minimal tree

V_1, V_2, V_3, V_4 are terminals V_5 and V_6 is non-terminals.

Definition 2.3

A class of problems solvable by non deterministic polynomial time algorithm is called **NP**.

Definition 2.4

A problem is **NP-complete** if

1. It is an element of the class NP.
2. Another NP-complete problem is polynomial time reducible to it.

Definition 2.5

A Steiner minimum tree for K is given such that some of the terminals are interior points. Then we decompose this tree into components so that terminals only occur as leaves of these components. Such a component is called **full component.**

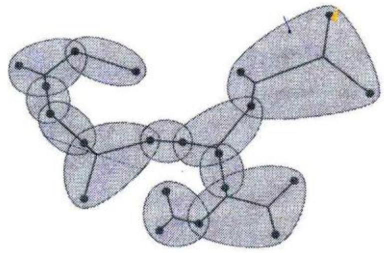

Figure 5: The full components of a Steiner Tree

Definition 2.6

The **Coxeter graph** is a non-hamiltonian cubic symmetric graph on 28 vertices and 42 edges.

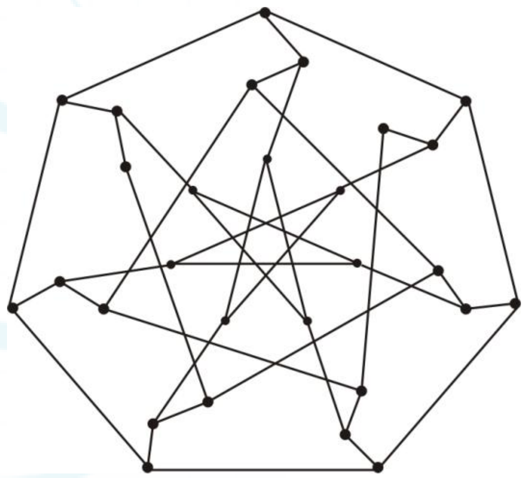

Figure 6: The Coxeter graph

Properties of the Coxeter graph

1. A spanning cycle in a graph is called a Hamiltonian cycle.
2. A graph having a Hamiltonian cycle is called a Hamiltonian graph.
3. The coxeter graph has no Hamiltonian cycle

This implies coxeter graph is not Hamiltonian.

4. A regular graph of degree 3 is called cubic graph.
5. Each vertex of a coxeter graph is of degree 3.

This implies coxeter graph is called cubic graph.

Definition 2.7

A symmetric graph is a graph in which both edge and vertex are transitive.

Definition 2.8

An edge transitive graph is a graph such that any two edges are equivalent under some element of its automorphism group.

Example

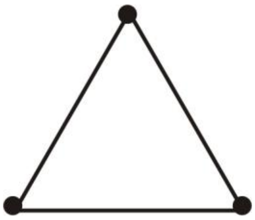

Figure 7: Triangle graph

More precisely, a graph is edge transitive if for all pairs of edges $(e_1 e_2)$ there exists an elements γ of the edge automorphism group $A_u t(G)$ such that $\gamma(e_1) = e_2$. An undirected graph is edge transitive iff its line graph is vertex transitive.

Definition 2.9

A **vertex – transitive graph** is a graph such that every pair of vertices is equivalent under some element of its automorphism graph.

Example

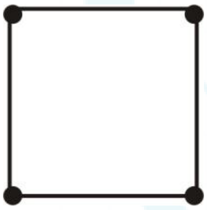

Figure 8: Square graph

Steiner Problem in Coxeter Graph Which is NP-Complete

Result 1

Steiner problem in Coxeter graph which is NP-complete.

Proof

Let the Steiner problem in graph is $\in$ NP, it is sufficient to show that Steiner problem in Coxeter graph is NP-complete.

To construct a Coxeter graph on 28 vertices and 42 edges.

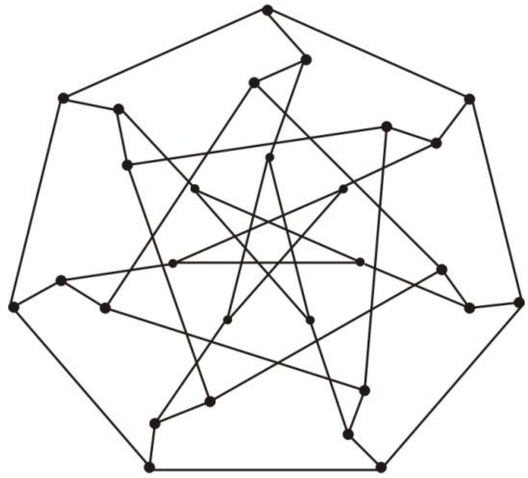

Figure 9: Coxeter graph

Now we take a Steiner tree from the Coxeter graph of maximum number of vertices in the following Figure 10.

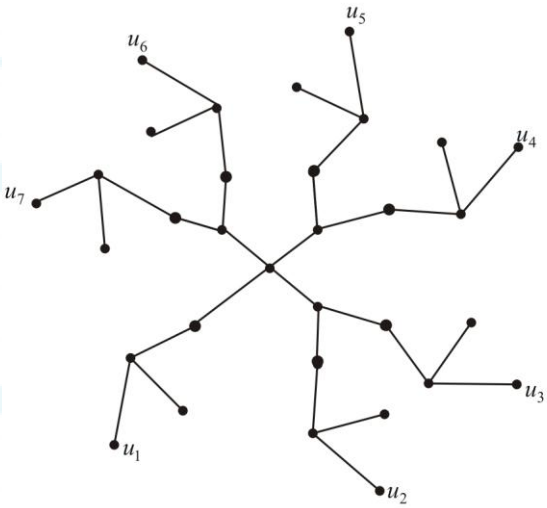

Figure 10: A Steiner tree in Coxeter graph

We reduce 3SAT to Steiner problem in Coxeter graphs. Let $x_1, x_2, x_3, \ldots, x_n$ be the variables $c_1, c_2, \ldots, c_n$ the clauses in an arbitrary instance of 3 SAT.

Our aim is to construct a Coxeter graph $G = (V, E)$ and a terminal set K, and a bound B such that Coxeter contains Steiner tree T for K at size at most B if and only if the given 3 SAT instance is satisfiable.

Transforming 3 SAT to Steiner problem in Coxeter graph is constructed as follows. We connect u_1 and u_7 by a variable path in Figure 11.

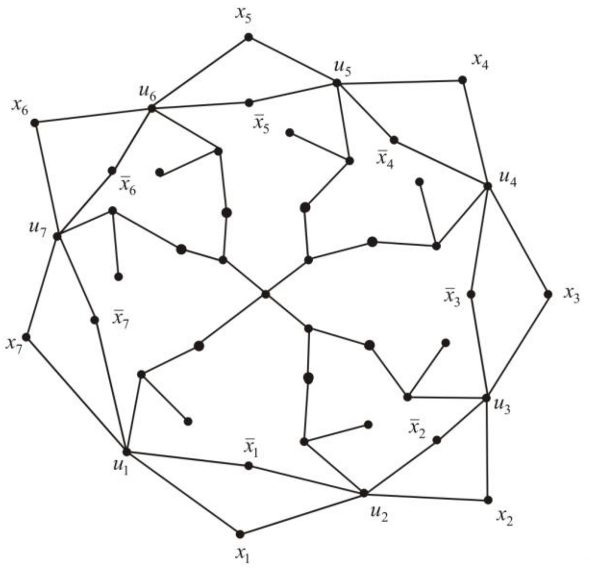

Figure 11: Transforming 3SAT to Steiner Problem in Coxeter graph

Then we crate for every clause gadget consisting of a vertex c_i that is connected to the literals contained in the clause c_i be paths of Length $t = 2n + 1$.

As terminal set $K = \{u_1 u_7\} \cup \{c_1, c_2, c_3, c_4\}$ and set $B = 2n + t.m$

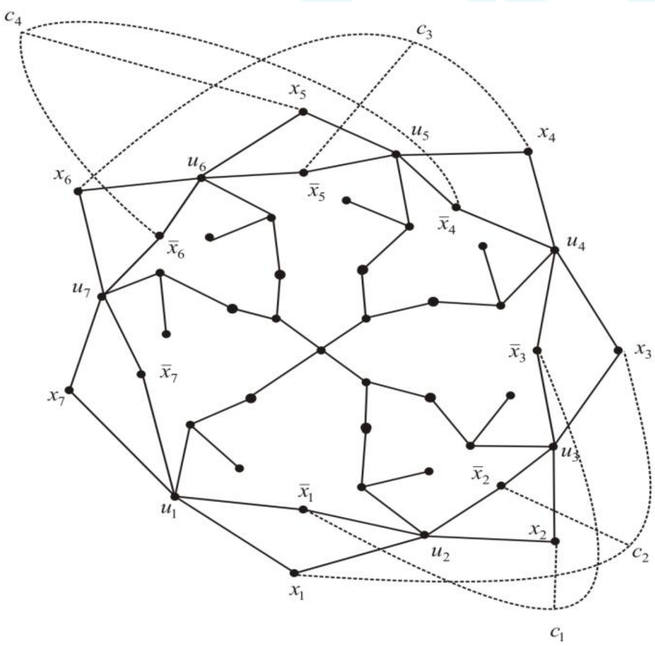

Figure 12: The clause gadget for the clauses

$$c_1 = \bar{x}_1 \vee x_2 \vee \bar{x}_3$$
$$c_2 = x_3 \vee \bar{x}_2 \vee x_1$$
$$c_3 = x_4 \vee \bar{x}_5 \vee x_6$$
$$c_4 = \bar{x}_4 \vee x_5 \vee \bar{x}_6$$

The dashed lines indicated paths of Length $t = 2n + 1$ from C_i to the appropriate vertices on the variable path.

Let $x_1 \in P$ if x_1 is said to true in this assignment and $\bar{x}_1 \in P$ otherwise. To construct a Steiner tree for K we start with $u_1 - u_7$ path reflecting a satisfying assignment.

x_1 is true for variables. Hence we arrive from our SAT problem seven variable and 4 clauses for a 3SAT problem.

The number of variables $n = 7$ to form the clauses $\{c_1, c_2, c_3, c_4\}$ and the terminal set $K = \{u, v\} \cup \{c_1, c_2, c_3, c_4\}$ and $B = 2n + t.m$

Next observe that every clause the vertex c_i can he connected by path of Length.

$$t = 2n + 1$$

$$n = 7 \Rightarrow t = 2(7) + 1 = 15$$

In this way we obtain a Steiner tree for K of Length $B = 2n + t.m$

$$m = 4 \Rightarrow \; B = 2n + t.m$$
$$= 2(7) + 15.4$$
$$= 14 + 60$$
$$= 74$$

A Steiner tree for this Coxeter graph we starting with a $u_1 - u_7$ path P reflecting a satisfying assignment.

On the other hand, we assume now that T is a Steiner tree for K of Length at most B, Trivially for each clause to the vertex c_i has to be connected to the variable path.

Then $|E(T)| \geq (m+1).t > B$
$$\geq (4+1).15 > 74$$
$$\geq 5.15 > 74$$
$$\geq 75 > 74 \Rightarrow \text{Contradiction}$$

In this graphs $u_1 - u_7$ path contains 28 edges and that each clause gadget is connected to this path using exactly t edges.

This shows that u_1 and u_7 can only be connected along the variable path, which requires atleast $2n$ edges.

In this Coxeter graph $u_1 - u_7$ Path contains 28 edges. Thus $u_1 - u_7$ path reflects a satisfying assignment. This implies that a Steiner problem in Coxeter graph is NP-complete.

In figure 10 the Steiner tree contains 14 terminals.

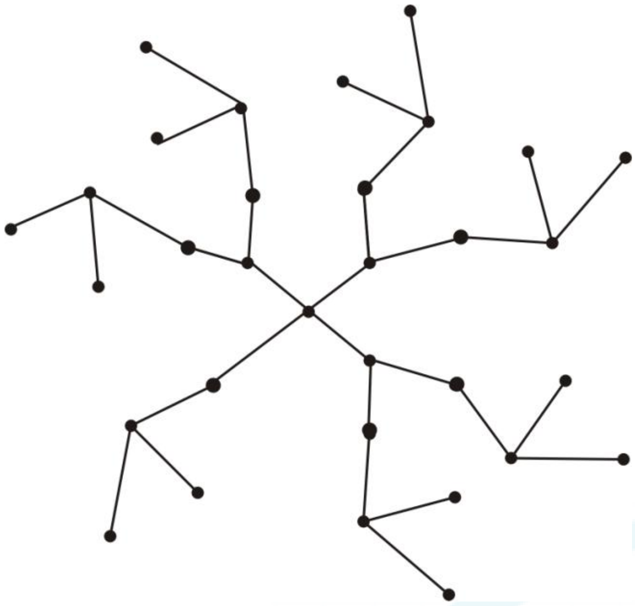

Figure 13: A Steiner tree in Coxeter graph

This implies that every full component of a Coxeter graph contains at most four terminals. This implies that R-Restricted Steiner problem in Coxeter graph is NP-complete.

Result 1

Steiner problem in Coxeter graph is NP-complete.

Result 2

Every $u_1 - u_7$ path of Steiner tree in Coxeter graph is NP-complete and every $u_1 - u_7$ path of Coxeter graph contains exactly $2n$ edges.

Result 3

A Steiner tree of Coxeter graph, every full component contains at most 4 terminals.

Result 4

Transforming 3SAT to Steiner problem in Coxeter graph is NP-complete.

3. Conclusion

In this paper, we proved that the Steiner tree problem in Coxeter graph which is NP-complete. We have also shown that every full component of a Steiner tree contains almost 4 terminals and every $u - v$ path of Steiner tree in Coxeter graph contains exactly 2n edges.

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