$$Q_1 = \tau - D_r\dot{\theta}$$

and acting on the pendulum is

$$Q_2 = -D_p\dot{\alpha}$$

The total potential energy of the system is

$$V = mg\frac{L_p}{2}\cos(\alpha)$$

and the total kinetic energy of the system is

$$T = \frac{1}{2}(J_r + m_p {L_r}^2)\dot{\theta}^2 + \frac{2}{3}m_p {L_p}^2\dot{\alpha}^2 - m_p L_p L_r \cos(\alpha)\dot{\theta}\dot{\alpha}$$

Solving the above two equations for the Lagrangian and the derivatives, the EOM of the system are

$$\left(m_p {L_r}^2 + \frac{1}{4}m_p {L_p}^2 - \frac{1}{4}m_p {L_p}^2\cos^2(\alpha) + J_r\right)\ddot{\theta} - \left(\frac{1}{2}m_p L_p L_r \cos(\alpha)\right)\ddot{\alpha} + \left(\frac{1}{2}m_p {L_p}^2 \sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2}m_p L_p L_r \sin(\alpha)\right)\dot{\alpha}^2 = \tau - D_r\dot{\theta}$$

$$\frac{1}{2}m_p L_r L_p \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4}m_p {L_p}^2\right)\ddot{\alpha} - \frac{1}{4}m_p {L_p}^2\cos(\alpha)\sin(\alpha)\dot{\theta}^2 + \frac{1}{2}m_p L_p g \sin(\alpha) = -D_p\dot{\alpha}$$

The torque applied at the base of the rotary arm is described as

$$\tau = \frac{\eta_g \eta_m k_g k_t k_m (V_m - k_m\dot{\theta})}{R_m}$$

When the nonlinear equations are linearized about the operating point$(\theta, \alpha) = (0,0)$, the resultant EMO of the inverted pendulum are defined as:

$$(m_p {L_r}^2 + J_r)\ddot{\theta} - \frac{1}{2}m_p L_p L_r\ddot{\alpha} = \tau - D_r\dot{\theta}$$

and

$$\frac{1}{2}m_p L_r L_p\ddot{\theta} + \left(J_p + \frac{1}{4}m_p {L_p}^2\right)\ddot{\alpha} + \frac{1}{2}m_p L_p g\alpha = -D_p\dot{\alpha}$$

Solving the above equations for the acceleration terms yields

$$\ddot{\theta} = \frac{1}{J_T}\left\{-\left(J_p + \frac{1}{4}m_p {L_p}^2\right)D_r\dot{\theta} + \frac{1}{2}m_p L_p L_r D_p\dot{\alpha} + \frac{1}{4}{m_p}^2{L_p}^2 L_r g\alpha + \left(J_p + \frac{1}{4}m_p {L_p}^2\right)\tau\right\}$$

and

$$\ddot{\alpha} = \frac{1}{J_T}\left\{\frac{1}{2}m_p L_p L_r D_r\dot{\theta} - (J_r + m_p {L_r}^2)D_p\dot{\alpha} - \frac{1}{2}m_p L_p g(J_r + m_p {L_r}^2)\alpha - \frac{1}{2}m_p L_p L_r\tau\right\}$$

Where

$$J_T = J_p m_p {L_r}^2 + J_r J_p + \frac{1}{4}J_r m_p {L_p}^2$$

The A and B matrices for state-space representation can then be found as

$$A = \frac{1}{J_T}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4}{m_p}^2{L_p}^2 L_r g & -\left(J_p + \frac{1}{4}m_p {L_p}^2\right)D_r & \frac{1}{2}m_p L_p L_r D_p \\ 0 & -\frac{1}{2}m_p L_p g(J_r + m_p {L_r}^2) & \frac{1}{2}m_p L_p L_r D_r & -(J_r + m_p {L_r}^2)D_p \end{bmatrix}$$

$$B = \frac{1}{J_T}\begin{bmatrix} 0 \\ 0 \\ J_p + \frac{1}{4}m_p {L_p}^2 \\ -\frac{1}{2}m_p L_p L_r \end{bmatrix}$$