

High Order Stable Methods Derived from the Composition of Reverse Adams Moulton and Adams Moulton Methods

Ajie I.J.¹, Durojaiye M.O.², Ikhile M.N.O.³, Onumanyi P.¹

¹National Mathematical Centre, Abuja, Nigeria

²University of Abuja, Nigeria

³University of Benin, Benin City, Nigeria

Abstract: In this work, a family of high order stable methods is derived for the solution of stiff ordinary differential equation. The derivation is done by pairing Reverse Adams Moulton (RAM) and Adams Moulton (AM) methods and applying shift-operator E on them. The resultant one-block methods are A -Stable for order ten and $A(\alpha)$ -stable with $\alpha = 74.60^\circ$ for order eleven. The methods are tested on some stiff initial value problems to showcase the effectiveness.

Keywords: Shift-operator, $A(\alpha)$ -stable, Stiff ordinary differential equation, Reverse Adams Moulton and Adams Moulton

1. Introduction

Many physical phenomena when modeled mathematically result in ordinary differential equations (ode). The solutions of these equations enable us to find answers to such questions as to how a physical system behaves. Most often, providing analytical solution to these modeled equations is very difficult, if not impossible and as a result numerical approximations are often sought. In this work, we consider the problem of providing methods for finding the numerical solution $y(t)$ to the initial value problems (ivp) in ode

$$\begin{aligned} y'(t) &= f(t, y(t)); & y(t_0) &= y_0; & t &\in [a, b]; \\ f &: \mathfrak{R} \times \mathfrak{R}^m \rightarrow \mathfrak{R}^m; & y &: \mathfrak{R} \rightarrow \mathfrak{R}^m \end{aligned} \quad (1)$$

Equation (1) occurs in many fields of science and engineering applications and therefore needs extensive study.

2. Linear Multistep Formulas (LMF)

The LMF is generally given by

$$\sum_{r=0}^k \alpha_r y_{n+r} = h \sum_{r=0}^k \beta_r f(t_{n+r}, y_{n+r}) \quad (2)$$

where the step number $k > 1$ and $h_n = t_{n+1} - t_n$ is a variable step length, $\{\alpha_r\}_{r=0}^k$ and $\{\beta_r\}_{r=0}^k$ are both not zero. Formula (2) can be represented by two polynomials

$$\rho(z) = \sum_{r=0}^k \alpha_r z^r, \quad \sigma(z) = \sum_{r=0}^k \beta_r z^r \quad (3)$$

such that (2) can be rewritten as

$$\rho(E)y_n = h\sigma(E)f_n \quad (4)$$

where E is the shift operator defined by $E^j y_n = y_{n+j}$.

3. Adams Moulton (AM) methods

If in (2) $\alpha_k = 1$ and $\alpha_{k-1} = -1$, and all other coefficients of $y(t)$ are zeros then the resultant formula is known as AM and is written as

$$y_{n+k} - y_{n+k-1} = h \sum_{r=0}^k \beta_r f(t_{n+r}, y_{n+r}) \quad (5)$$

The first characteristics polynomial of (5) is $\rho(z) = z^{k-1}(z-1)$. By this, the methods are zero-stable. The determination of the coefficients $\{\beta_r\}_{r=0}^k$ is done by imposing the maximum order $p = k + 1$. This leads to

$$\begin{pmatrix} 1 \\ \frac{k^2 - (k-1)^2}{2} \\ \frac{k^3 - (k-1)^3}{3} \\ \cdot \\ \cdot \\ \cdot \\ \frac{k^{k+1} - (k-1)^{k+1}}{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & 3 & \dots & k \\ 0 & 1 & 2^2 & 3^2 & \dots & k^2 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 1 & 2^k & 3^k & \dots & k^k \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_k \end{pmatrix} \quad (6)$$

If on the other hand, $\alpha_1 = 1$ and $\alpha_0 = -1$ and all other coefficients of $y(t)$ are zeros then we have the Reversed Adams Moulton (RAM) methods which are generally written as

$$y_{n+1} - y_n = h_n \sum_{r=0}^k \beta_r f_{n+r} \quad (7)$$

Their first characteristics polynomial $\rho(z)$ are given by $\rho(z) = z - 1$ and are therefore are generally zero stable. The coefficients $\{\beta_r\}_{r=0}^k$ are determination by imposing the maximum order $k + 1$ on the method (7). This leads to the matrix equations

$$\begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & 3 & \dots & k \\ 0 & 1 & 2^2 & 3^2 & \dots & k^2 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 1 & 2^k & 3^k & \dots & k^k \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_k \end{pmatrix} \quad (8)$$

which are solved simultaneously for the coefficients [7].

Stiff ode can only be handled effectively with A-stable method but A-stable linear multistep methods (LMM) are difficult to come by because of Dahlquist order barrier. The order barrier theorem of Dahlquist [9] states that:

Theorem: Dalquist order barrier

- (i) No explicit LMM can be A-stable;
- (ii) No A-stable LMM can have order greater than two;
- (ii) The second order A-stable LMM with the smallest error constant is the

$$\text{trapezoidal rule, } y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1}); C_{p+1} = \frac{1}{12}, p = 2$$

where C_{p+1} is the error constant and p is the order

Definition 1

A LMM is said to be A-Stable if its region of absolute stability (RAS) contains the entire negative (left) complex half-plane \mathcal{C} , (see [14]).

In [10], it was noted that the above Dalquist order barrier theorem on A-Stability of LMM forced many researchers to considered $A(\alpha)$ -Stable methods and also adopt some unconventional numerical integrators. See the following ([1], [2], [3], [4], [5], [6], [7], [8], [15] and [16]) for some of the authors that considered unconventional numerical integrators in order to circumvent the order barrier. In order to circumvent the barrier theorem, shift operator is used in this work to transform the LMM pairs into a family of one-step block integrators that are $A(\alpha)$ – Stable at high order which are suitable for solving stiff ode.

Definition 2:

A LMM when applied to a linear test equation $y' = \lambda y$ is said to be $A(\alpha)$ – Stable, with $\alpha \in (0, \frac{\pi}{2})$ if its region of absolute stability (RAS) contains the infinite wedge W_α ,

$$W_\alpha = \left\{ \lambda h : -\alpha \leq \left| \pi - \arg(\lambda h) \right| \leq \alpha \right\} \quad (9)$$

The LMM becomes A-Stable when $\alpha = \frac{\pi}{2}$, (see [20]).

4. Derivation of the block methods

The methodology for the construction is captured in the following proposition:

Proposition

Let the multi-family of LMM $\left\{ \rho_k^{[j]}(R), \sigma_k^{[j]}(R) \right\}_{j=1, k=1}^{m, K}$ be given, that is,

$$\rho_k^{[j]}(E) y_n = h \sigma_k^{[j]}(E) f_n ; j = 1(1)m, k = 1(1)K \quad (10)$$

with $\left\{ \rho_k^{[j]}, \sigma_k^{[j]} \right\}$ for a fixed j forming a family of variable order $P_{k, j}$ of variable step number k . Then the resultant system of composite LMM

$$E^i \rho_k^{[j]}(E) y_n = h E^i \sigma_k^{[j]}(E) f_n ; i = 0(1)k - l ; j = 1, 2, \dots, m \quad (11)$$

arising from the E-operator transformation of (10) can be composed as the block method

$$A_1 Y_{n+1} + A_0 Y_n = h(B_1 F_{n+1} + B_0 F_n) ; \det(A_1) \neq 0 \quad (12)$$

if k is chosen such that l is an integer given as

$$l = \frac{m + k(m - 2)}{m - 1} ; m, k \geq 2 \text{ and } k - l \geq 0. \quad (13)$$

Consider the k-step LMF pair defined by $[\rho_1, \sigma_1]$ and $[\rho_2, \sigma_2]$. Shifting this (k-1) time, where l is as defined in (13), we have a set of 2(k-1) equations in 2(k-1) unknowns which can be written in the block form (12). Note that $Y_n \cap Y_{n+1} = \phi$ (empty) and $F_n \cap F_{n+1} = \phi$. Equation (12) can also be multiplied by A_1^{-1} and written in the form

$$Y_{n+1} = A_1^{-1}A_0Y_n + h(A_1^{-1}B_1F_{n+1} + A_1^{-1}B_0F_n) \quad (16)$$

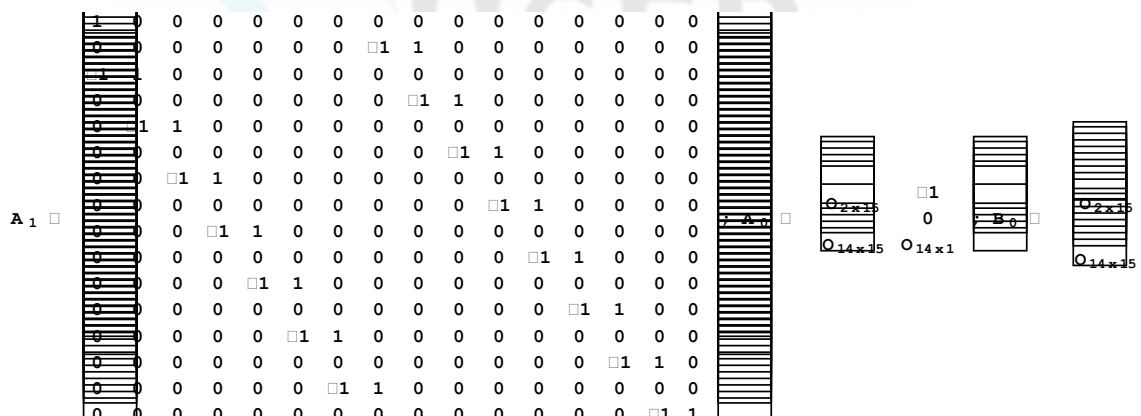
where

$$A_1^{-1}A_0 = \left(\begin{array}{c|c} & \begin{matrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{matrix} \\ \hline O & \cdot \end{array} \right) = (O \quad e) ; \quad A_1^{-1}B_0 = \left(\begin{array}{c|c} & \begin{matrix} \times \\ \times \\ \cdot \\ \cdot \\ \cdot \\ \times \end{matrix} \\ \hline O & \cdot \end{array} \right) \quad \text{or} \quad A_1^{-1}B_0 = O$$

$$A_1^{-1}B_1 = \left(\begin{array}{cccccc} \times & \cdot & \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot & \times \end{array} \right) ;$$

The structure of $A_1^{-1}A_0$ and $A_1^{-1}B_0$ makes the need to carry along the past solution $Y_n = (y_{n-2k+l+1}, y_{n-2k+l+2}, \dots, y_{n-1}, y_n)^T$; unnecessary. Only the initial solution y_n provided by the ODE (1) is needed to implement the block method (12). Therefore (12) is self-starting block methods.

The coefficients of the order nine of the methods are given here below:



9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0	0	0	0	0	0
907200	907200	90720	3628800	3628800	453600	907200	1451520	1036800						
335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0	0	0	0	0	0
907200	453600	3628800	3628800	90720	907200	7257600	89600							
9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0	0	0	0	0	0
7257600	907200	90720	3628800	3628800	453600	907200	1451520	1036800						
116687	335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0	0	0	0	0
1451520	907200	453600	3628800	3628800	90720	907200	7257600	89600						
25713	9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0	0	0	0	0
89600	7257600	907200	90720	3628800	3628800	453600	907200	1451520	1036800					
8183	116687	335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0	0	0	0
1036800	1451520	907200	453600	3628800	3628800	90720	907200	7257600	89600					
0	25713	9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0	0	0	0
0	89600	7257600	907200	90720	3628800	3628800	453600	907200	1451520	1036800				
0	8183	116687	335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0	0	0
0	1036800	1451520	907200	453600	3628800	3628800	90720	907200	7257600	89600				
0	0	25713	9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0	0	0
0	89600	7257600	907200	90720	3628800	3628800	453600	907200	1451520	1036800				
0	0	8183	116687	335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0	0
0	0	1036800	1451520	907200	453600	3628800	3628800	90720	907200	7257600	89600			
0	0	0	25713	9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0	0
0	0	0	89600	7257600	907200	90720	3628800	3628800	453600	907200	1451520	1036800		
0	0	0	8183	116687	335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0
0	0	0	1036800	1451520	907200	453600	3628800	3628800	90720	907200	7257600	89600		
0	0	0	0	25713	9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0
0	0	0	89600	7257600	907200	90720	3628800	3628800	453600	907200	1451520	1036800		
0	0	0	8183	116687	335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0
0	0	0	1036800	1451520	907200	453600	3628800	3628800	90720	907200	7257600	89600		
0	0	0	0	25713	9449717	1408913	200029	8641823	6755041	462127	335983	116687	8183	0
0	0	0	89600	7257600	907200	90720	3628800	3628800	453600	907200	1451520	1036800		
0	0	0	8183	116687	335983	462127	6755041	8641823	200029	1408913	9449717	25713	0	0
0	0	0	1036800	1451520	907200	453600	3628800	3628800	90720	907200	7257600	89600		

5. Stability of the Implicit Block Methods

When (12) is applied to the test equation

$$y' = \lambda y, \text{Re}(\lambda) < 0 \quad (17)$$

it yields the characteristics equation.

$$\pi(w, z) = \det(A_1 w + A_0 - z(B_1 w + B_0)), \quad z = \lambda h \quad (18)$$

The region of absolute stability R_A associated with (12) is the set

$$R_A = \{z : |w_j(z)| \leq 1, j = 1(1)k\} \quad (19)$$

If we let $z \rightarrow 0$ in (18), the difference system becomes

$$\pi(w, 0) = \det(A_1 w + A_0) \quad (20)$$

All the proposed block methods can be cast in the form

$$A_1 Y_{n+1} + \hat{a} y_n = h(B_1 F_{n+1} + \hat{b} f_n) \quad (21)$$

Where

$$\hat{a} = \begin{pmatrix} \alpha_0^{[1]} \\ \alpha_0^{[2]} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_0^{[m]} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}_{2(k-l) \times 1}; \quad \hat{b} = \begin{pmatrix} \beta_0^{[1]} \\ \beta_0^{[2]} \\ \cdot \\ \cdot \\ \cdot \\ \beta_0^{[m]} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}_{2(k-l) \times 1} \quad (22)$$

Notice that for all the block methods, $A_1^{-1}\hat{a} = (1\ 1\ 1\dots 1)^T = e$

$$A_1^{-1}A_0 = \left(\begin{array}{c|c} & 1 \\ & 1 \\ O & \cdot \\ & \cdot \\ & \cdot \\ & 1 \end{array} \right) = \left(\begin{array}{c|c} & \\ O & e \end{array} \right)$$

implying that

To see this, assume order $p \geq 1$ for all the LMF that constitute the block, then by consistency,

$$A_1 e + \hat{a} = 0 \quad (23)$$

where $e = (1\ 1\ 1\dots 1)^T$. From (21) it follows that

$$A_1^{-1}\hat{a} = -e \quad (24)$$

The above ensures zero-stability of the implicit block methods (12). Method (12) can also be written as

$$Y_{n+1} = M(z)Y_n \quad (25)$$

where

$$M(z) = (I - zA_1^{-1}B_1)^{-1}(zA_1^{-1}B_0 - A_1^{-1}A_0) \quad (26)$$

is the amplification matrix. The stability function $p(w, z)$ is

$$p(w, z) = \text{Det}[I_k w - M(z)] = w^{k-1}(w - D(z)) \quad (27)$$

The stability domain S of this family is

$$S = \{z \in \mathbb{C} : |w(z)| \leq 1\} \quad (28)$$

The $D(z)$ (the only non-zero value of $R(z)$) for this family of methods are given as a rational function $D(z) = \frac{P(z)}{Q(z)}$.

where $P(z)$ and $Q(z)$ are polynomials.

Definition1: A block method is said to be pre-stable if the roots of $Q(z)$ are contained in \mathbb{C}^+ (see [7]).

The one step block method is A-stable if and only if it is stable on the imaginary axis (I-stable): $D(iy) \leq 1$ for all $y \in \mathbb{R}$, and $D(z)$ is analytic for $\text{Re}(z) < 0$ (i.e., $Q(z)$ does not have roots with negative or zero real parts), I-stability is equivalent to the fact that the Norsett polynomial defined by

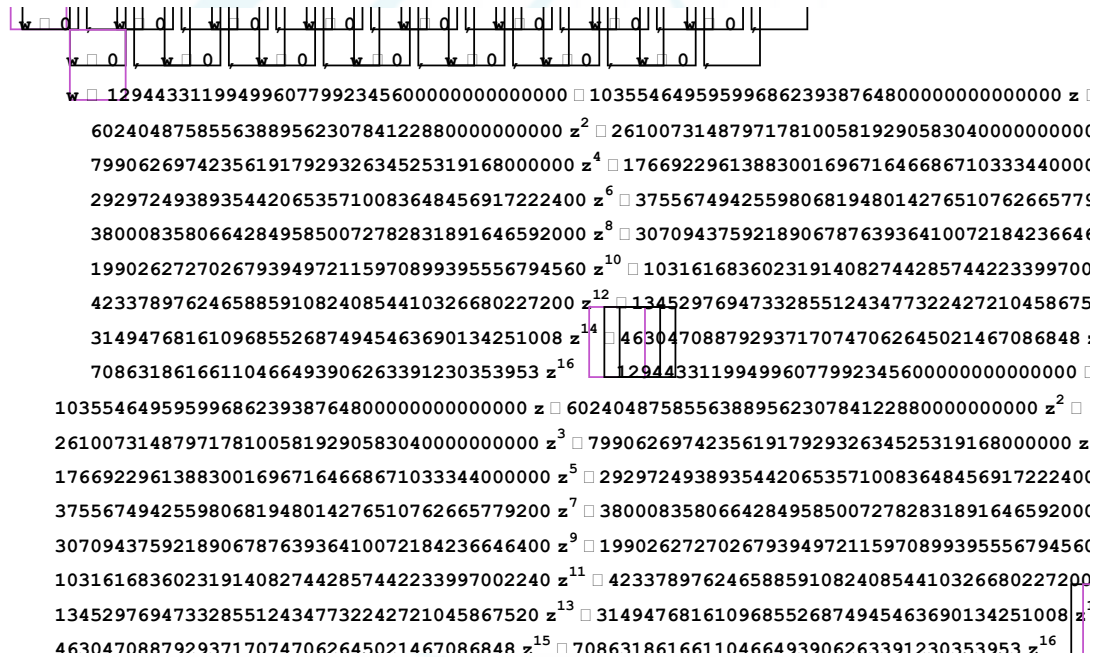
$$E(y) = |Q(iy)|^2 - |P(iy)|^2 = Q(iy)Q(-iy) - P(iy)P(-iy) \quad (29)$$

satisfies $E(y) > 0$ for all $y \in \mathbb{R}$ [13].

For this particular case of order 10, the characteristics equation $\pi(w, z)$ in (18) is given by

$$w^{14} + 12944331199499607799234560000000000000000 w^{13} + 1035546495996862393876480000000000000000 w^{12} + 6024048758556388956230784122880000000000000000 w^{11} + 261007314879717810058192905830400000000000000000 w^{10} + 799062697423561917929326345253191680000000 w^9 + 176692296138830016967164668671033344000000 w^8 + 292972493893544206535710083648456917222400 w^7 + 375567494255980681948014276510762665779200 w^6 + 380008358066428495850072782831891646592000 w^5 + 307094375921890678763936410072184236646400 w^4 + 199026272702679394972115970899395556794560 w^3 + 103161683602319140827442857442233997002240 w^2 + 42337897624658859108240854410326680227200 w + 13452976947332855124347732242721045867520 z^{14} + 13452976947332855124347732242721045867520 z^{13} + 3149476816109685526874945463690134251008 z^{12} + 463047088792937170747062645021467086848 z^{11} + 70863186166110466493906263391230353953 z^{10} + 1035546495996862393876480000000000000000 z^9 + 6024048758556388956230784122880000000000000000 z^8 + 261007314879717810058192905830400000000000000000 z^7 + 799062697423561917929326345253191680000000 z^6 + 17669229613883001696716466867103334400000 z^5 + 292972493893544206535710083648456917222400 z^4 + 375567494255980681948014276510762665779200 z^3 + 380008358066428495850072782831891646592000 z^2 + 30709437592189067876393641007218423664640 z + 199026272702679394972115970899395556794560 z^{12} + 103161683602319140827442857442233997002240 z^{11} + 42337897624658859108240854410326680227200 z^{10} + 134529769473328551243477322427210458675 z^9 + 3149476816109685526874945463690134251008 z^8 + 463047088792937170747062645021467086848 z^7 + 70863186166110466493906263391230353953 z^6 + 1035546495996862393876480000000000000000 z^5 + 6024048758556388956230784122880000000000000000 z^4 + 261007314879717810058192905830400000000000000000 z^3 + 79906269742356191792932634525319168000000 z^2 + 17669229613883001696716466867103334400000 z + 292972493893544206535710083648456917222400 z^{12} + 375567494255980681948014276510762665779200 z^{11} + 380008358066428495850072782831891646592000 z^{10} + 30709437592189067876393641007218423664640 z^9 + 199026272702679394972115970899395556794560 z^8 + 103161683602319140827442857442233997002240 z^7 + 42337897624658859108240854410326680227200 z^6 + 13452976947332855124347732242721045867520 z^5 + 3149476816109685526874945463690134251008 z^4 + 463047088792937170747062645021467086848 z^3 + 70863186166110466493906263391230353953 z^2 + 1035546495996862393876480000000000000000 z + 6024048758556388956230784122880000000000000000$$

The roots of this characteristic equation is



For this particular case, the only none zero solution $D(z)$, has no pole on C^- , all the roots of $Q(z)$ which is the denominator of the rational function $D(z)$ are contained in C^+ as shown below

$$z = 9.3622e-12-0.2518i, z = 9.3622e-12+0.2518i, z = 0.0991 -1.8292i, z = 0.0991 +1.8292i, z = 0.1453 -4.3834i, z = 0.1453 +4.3834i, z = 0.3053 -1.1253i, z = 0.3053 +1.1253i, z = 0.5462 - 0.843i, z = 0.5462+0.8433i, z = 0.6585 -0.6008i, z = 0.6585 + 0.6008i, z = 0.7343 - 0.3490i, z = 0.7343 + 0.3490i, z = 0.7786 -0.1134i, z = 0.7786 + 0.1134i$$

$E(y) > 0_{in(29)}$ for all $y \in \mathfrak{R}$, the method is therefore A-Stable

6. Numerical Experiments

In this section, we considered two problems to test the effectiveness of the method

Problem 1: Singularly Perturbed Problem (cf: [12])

$$y_1'(t) = -(2 + \varepsilon^{-1})y_1(t) + \varepsilon^{-1}y_2^2(t); y_1(0) = 1$$

$$y_2'(t) = y_1(t) - y_2(t) - y_2^2(t); y_2(0) = 1$$

$$t \in [0,1]; \varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$$

The theoretical solutions are $y_1(t) = e^{-2t}$; $y_2(t) = e^{-t}$. The errors when our method of order 9 is implemented on problem 1 are given in table 1. Even when the value of ε is tending to zero, the errors are consistent.

Table 1: errors on different vales of ε on problem 2 using RAM/AM k=9

h=0.0001	Errors
$\varepsilon = 10^{-4}$	3.58e-08
$\varepsilon = 10^{-3}$	6.29e-07
$\varepsilon = 10^{-2}$	4.05e-07
$\varepsilon = 10^{-1}$	5.01e-08

Problem 2:

Consider the following linear constant coefficient initial value problem taken from [7],

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y; \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The theoretical solution is given by

$$y(t) = \frac{1}{2} \begin{pmatrix} e^{-2t} + e^{-40t} (\cos(40t) + \sin(40t)) \\ e^{-2t} - e^{-40t} (\cos(40t) + \sin(40t)) \\ 2e^{-40t} (\sin(40t) - \cos(40t)) \end{pmatrix}$$

The errors and computational rate of convergence computed with our method of order 10 at different values of h are given in table 2

Table 2: errors and convergence rate of RAM/AM k=9 on problem 2

h	Errors	Convergence rate
0.032	1.70e-02	
0.016	2.06e-04	6.3650
0.008	1.02e-06	7.6567
0.004	3.61e-09	8.1409
0.002	4.55e-12	9.6330
0.001	7.35e-15	9.2754

7. Conclusion

The work in [1] has been extended to higher order methods. The order ten of the family is $A - Stable$ while the order eleven is $A(\alpha) - Stable$ with $\alpha = 74.5934^\circ$. The errors from the implementation of the method of order ten on some stiff initial value problems shows that they are effective.

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