High Order Stable Methods Derived from the Composition of Reverse Adams Moulton and Adams Moulton Methods

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Abstract: ***In this work, a family of high order stable methods is derived for the solution of stiff ordinary differential equation. The derivation is done by pairing Reverse Adams Moulton (RAM) and Adams Moulton (AM) methods and applying shift-operator E on them. The resultant one-block methods are A-Stable for order ten and $A(\alpha)$-stable with $\alpha = 74.60^o$ for order eleven. The methods are tested on some stiff initial value problems to showcase the effectiveness.***

Keywords: Shift-operator, $A(\alpha)$-stable, Stiff ordinary differential equation, Reverse Adams Moulton and Adams Moulton

1. Introduction

Many physical phenomena when modeled mathematically result in ordinary differential equations (ode). The solutions of these equations enable us to find answers to such questions as to how a physical system behaves. Most often, providing analytical solution to these modeled equations is very difficult, if not impossible and as a result numerical approximations are often sought. In this work, we consider the problem of providing methods for finding the numerical solution $y(t)$ to the initial value problems (ivp) in ode

$$\begin{aligned} &y'(t) = f(t, y(t)); \quad y(t_0) = y_0; \quad t \in [a,b]; \\ &f : \Re \times \Re^m \to \Re^m; \quad y : \Re \to \Re^m \end{aligned} \tag{1}$$

Equation (1) occurs in many fields of science and engineering applications and therefore needs extensive study.

2. Linear Multistep Formulas (LMF)

The LMF is generally given by

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h \sum_{r=0}^{k} \beta_r f(t_{n+r}, y_{n+r}) \quad (2)$$

where the step number $k > 1$ and $h_n = t_{n+1} - t_n$ is a variable step length, $\{\alpha_r\}_{r=0}^{k}$ and $\{\beta_r\}_{r=0}^{k}$ are both not zero. Formula (2) can be represented by two polynomials

$$\rho(z) = \sum_{r=0}^{k} \alpha_r z^r, \quad \sigma(z) = \sum_{r=0}^{k} \beta_r z^r \quad (3)$$

such that (2) can be rewritten as

$$\rho(E) y_n = h\sigma(E) f_n \quad (4)$$

where E is the shift operator defined by $E^j y_n = y_{n+j}$.

3. Adams Moulton (AM) methods

If in (2) $\alpha_k = 1$ and $\alpha_{k-1} = -1$, and all other coefficients of y(t) are zeros then the resultant formula is known as AM and is written as

$$y_{n+k} - y_{n+k-1} = h\sum_{r=0}^{k} \beta_r f(t_{n+r}, y_{n+r}) \qquad (5)$$

The first characteristics polynomial of (5) is $\rho(z) = z^{k-1}(z-1)$. By this, the methods are zero-stable. The determination of the coefficients $\{\beta_r\}_{r=0}^{k}$ is done by imposing the maximum order $p = k+1$. This leads to

$$\begin{pmatrix} 1 \\ \frac{k^2-(k-1)^2}{2} \\ \frac{k^3-(k-1)^3}{3} \\ . \\ . \\ . \\ \frac{k^{k+1}-(k-1)^{k+1}}{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1.\,.\,.1 \\ 0 & 1 & 2 & 3.\,.\,.k \\ 0 & 1 & 2^2 & 3^2.\,.\,.k^2 \\ . & . & . & .\,.\,.\,.\,. \\ . & . & . & .\,.\,.\,.\,. \\ . & . & . & .\,.\,.\,.\,. \\ 0 & 1 & 2^k & 3^k.\,.\,.k^k \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_k \end{pmatrix} \qquad (6)$$

If on the other hand, $\alpha_1 = 1$ and $\alpha_0 = -1$ and all other coefficients of y(t) are zeros then we have the Reversed Adams Moulton (RAM) methods which are generally written as

$$y_{n+1} - y_n = h_n \sum_{r=0}^{k} \beta_r f_{n+r} \qquad (7)$$

Their first characteristics polynomial $\rho(z)$ are given by $\rho(z) = z-1$ and are therefore are generally zero stable. The coefficients $\{\beta_r\}_{r=0}^{k}$ are determination by imposing the maximum order $k+1$ on the method (7). This leads to the matrix equations

$$\begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ . \\ . \\ . \\ \frac{1}{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1.\,.\,.1 \\ 0 & 1 & 2 & 3.\,.\,.k \\ 0 & 1 & 2^2 & 3^2.\,.\,.k^2 \\ . & . & . & .\,.\,.\,.\,. \\ . & . & . & .\,.\,.\,.\,. \\ . & . & . & .\,.\,.\,.\,. \\ 0 & 1 & 2^k & 3^k.\,.\,.k^k \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_k \end{pmatrix} \qquad (8)$$

which are solved simultaneously for the coefficients [7].

Stiff ode can only be handled effectively with A-stable method but A-stable linear multistep methods (LMM) are difficult to come by because of Dahlquist order barrier. The order barrier theorem of Dahlquist [9] states that:

Theorem: Dalquist order barrier

(i) No explicit LMMcan be A-stable;
(ii) No A-stable LMM can have order greater than two;
(ii) The second order A-stable LMM with the smallest error constant is the
trapezoidal rule, $y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1});\ C_{p+1} = \frac{1}{12}, p = 2$
where C_{p+1} is the error constant and p is the order

Definition 1

A LMM is said to be A-Stable if its region of absolute stability (RAS) contains the entire negative (left) complex half-plane$\mathbb{C}$, (see [14]).

In [10], it was noted that the above Dalquist order barrier theorem on A-Stability of LMM forced many researchers to considered A(α)-Stable methods and also adopt some unconventional numerical integrators. See the following ([1], [2], [3], [4], [5], [6], [7], [8], [15] and [16]) for some of the authors that considered unconventional numerical integrators in order to circumvent the order barrier. In order to circumvent the barrier theorem, shift operator is used in this work to transform the LMM pairs into a family of one-step block integrators that are $A(\alpha) - Stable$ at high order which are suitable for solving stiff ode.

Definition 2:

A LMM when applied to a linear test equation $y' = \lambda y$ is said to be $A(\alpha) - Stable$, with $\alpha \in (0, \frac{\pi}{2})$ if its region of absolute stability (RAS) contains the infinite wadge w_α ,

$$w_\alpha = \{\lambda h : -\alpha \leq |\pi - \arg(\lambda h)| \leq \alpha\ \} \quad (9)$$

The LMM becomes A-Stable when $\alpha = \frac{\pi}{2}$, (see [20]).

4. Derivation of the block methods

The methodology for the construction is captured in the following proposition:

Proposition

Let the multi-family of LMM $\left\{\rho_k^{[j]}(R), \sigma_k^{[j]}(R)\right\}_{j=1,k=1}^{m,K}$ be given, that is,

$$\rho_k^{[j]}(E) y_n = h\, \sigma_k^{[j]}(E) f_n\,;\ \ j = 1(1)m,\ \ k = 1(1)K \quad (10)$$

with $\left\{\rho_k^{[j]}, \sigma_k^{[j]}\right\}$ for a fixed j forming a family of variable order $p_{k,j}$ of variable step number k. Then the resultant system of composite LMM

$$E^i \rho_k^{[j]}(E) y_n = hE^i \sigma_k^{[j]}(E) f_n\ ;\ \ i = 0\,(1)\,k - l\,;\quad j = 1, 2, \ldots, m \quad (11)$$

arising from the E-operator transformation of (10) can be composed as the block method

$$A_1 Y_{n+1} + A_0 Y_n = h(B_1 F_{n+1} + B_0 F_n)\,;\ \det(A_1) \neq 0 \quad (12)$$

if k is chosen such that l is an integer given as

$$l = \frac{m + k(m-2)}{m-1}\ ;\ m, k \geq 2 \text{ and } k - l \geq 0. \quad (13)$$

where Y_{n+1}, Y_n; F_{n+1} *and* F_n $n = 0,1,2,\ldots$ are as defined below and A_1, A_0, B_1, B_0 are square matrices also defined below for a fixed m.

$$A_0 = \begin{pmatrix} & \alpha_0^{[1]} \\ & \alpha_0^{[2]} \\ & \cdot \\ & \cdot \\ & \cdot \\ & \alpha_0^{[m]} \\ O & 0 \\ & \cdot \\ & \cdot \\ & \cdot \\ & 0 \end{pmatrix}_{(2k-l)\times(2k-l)} ; \quad B_0 = \begin{pmatrix} & \beta_0^{[1]} \\ & \beta_0^{[2]} \\ & \cdot \\ & \cdot \\ & \cdot \\ & \beta_0^{[m]} \\ O & 0 \\ & \cdot \\ & \cdot \\ & \cdot \\ & 0 \end{pmatrix}_{(2k-l)\times(2k-l)} \qquad (14)$$

$$A_1 = \begin{pmatrix} \alpha_1^{[1]} & \cdot & \cdot & \cdot & \alpha_k^{[1]} & 0 & 0 & 0 & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & 0 \\ \alpha_1^{[m]} & \cdot & \cdot & \cdot & \alpha_k^{[m]} & 0 & \cdot & \cdot & & & & & 0 \\ \alpha_0^{[1]} & \alpha_1^{[1]} & \cdot & \cdot & \cdot & \alpha_k^{[1]} & \cdot & \cdot & & & & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & 0 & & & & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & & & & & & \cdot \\ \alpha_0^{[m]} & \alpha_1^{[m]} & & & & \alpha_k^{[m]} & 0 & & & & & & \cdot \\ 0 & \alpha_0^{[1]} & \alpha_1^{[1]} & & & & \alpha_k^{[1]} & & & & & & \cdot \\ 0 & \cdot & \cdot & & & \cdot & \cdot & & & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\ & \alpha_0^{[m]} & \alpha_1^{[m]} & \cdot & \cdot & \cdot & \alpha_k^{[m]} & 0 & & & & & \\ \cdot & \cdot & \cdot & & & & \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & & \cdot & \cdot & & & 0 \\ \cdot & & & & & & & & \cdot & \alpha_0^{[1]} & & & \alpha_k^{[1]} \\ & & & & & & & & \cdot & \cdot & & & \cdot \\ 0 & \cdot & \cdot & \cdot & & & & & 0 & \alpha_0^{[m]} & \cdot & \cdot & \alpha_k^{[m]} \end{pmatrix}_{(2k-l)\times(2k-l)}$$

$$B_1 = \begin{pmatrix}
\beta_1^{[1]} & . & . & . & \beta_k^{[1]} & 0 & 0 & 0 & . & . & . & & & & 0 \\
. & & & & & & & & & & & & & & \\
. & . & . & . & . & . & . & . & & & & & & & 0 \\
. & & & & & & & & & & & & & & \\
\beta_1^{[m]} & . & . & . & \beta_k^{[m]} & 0 & . & . & & & & & & & 0 \\
\beta_0^{[1]} & \beta_1^{[1]} & . & . & . & \beta_k^{[1]} & . & . & & & & & & & 0 \\
. & . & . & . & & . & 0 & & & & & & & & 0 \\
. & . & & & & . & & & & & & & & & \\
. & . & . & . & . & . & 0 & & & & & & & & . \\
\beta_0^{[m]} & \beta_1^{[m]} & & & & \beta_k^{[m]} & 0 & & & & & & & & . \\
0 & \beta_0^{[1]} & \beta_1^{[1]} & & & \beta_{k-1}^{1} & \beta_k^{[1]} & & & & & & & & . \\
0 & . & . & & & . & . & & & & & & & & \\
& . & . & . & . & . & . & & & & & & & & \\
& \beta_0^{[m]} & \beta_1^{[m]} & . & . & \beta_{k-1}^{[m]} & \beta_k^{[m]} & 0 & & & & & & & \\
& . & . & & & & . & & & & & & & & \\
. & . & . & & & & . & & & & & . & . & . & \\
. & . & . & & & & . & & & . & . & & & & 0 \\
. & & & & & & & & & . & \beta_0^{[1]} & & & & \beta_k^{[1]} \\
& & & & & & & & & . & . & & & & . \\
0 & . & . & . & & & & & & 0 & \beta_0^{[m]} & . & . & . & \beta_k^{[m]}
\end{pmatrix}_{(2k-l)\times(2k-l)}$$

$$Y_{n+1} = (y_{n+1}, y_{n+2}, \ldots, y_{n+2k-l})^T; \quad Y_n = (y_{n-2k+l+1}, y_{n-2k+l+2}, \ldots, y_{n-1}, y_n)^T; \tag{15}$$

$$F_{n+1} = (f_{n+1}, f_{n+2}, \ldots, f_{n+2k-l})^T; \quad F_n = (f_{n-2k+l+1}, f_{n-2k+l+2}, \ldots, f_{n-1}, f_n)^T$$

$n = 0, 1, 2, \ldots$

Proof:

Notice that the E-operator is effectively applied k-l times on the system of LMF $\left\{\rho_k^{[j]}, \sigma_k^{[j]}\right\}_{k,j}$. Thus there are 2k-l unknown solution points captured in the block of solution $Y_{n+1} = (y_{n+1}, y_{n+2}, \ldots, y_{n+2k-l})^T$. By this the block definition in (12) is realized if the coefficient matrices A_1, A_0, B_1, B_0 are square matrices of dimension $(2k-l)\times(2k-l)$. This simply imply that $m + m(k-l) = 2k-l$ so that l is as in (13) and for a fixed m the k is then chosen such that $k - l \geq 0$

In particular:

(1.) $m = 2$; $l = 2$ $k = 2, 3, 4, \ldots$

(2.) $m = 3$; $l = \dfrac{k+3}{2}$; $k = 3, 5, 7, \ldots$

(3.) $m = 4$; $l = \dfrac{4+2k}{3}$; $k = 4, 7, 10, 13, \ldots$

When $k - l = 0$, the method is a minimal block method. This is so if $m = k$. However, the case of interest in this paper is when m = 2.

Consider the k-step LMF pair defined by $[\rho_1, \sigma_1]$ and $[\rho_2, \sigma_2]$. Shifting this (k-l) time, where l is as defined in (13), we have a set of 2(k-1) equations in 2(k-1) unknowns which can be written in the block form (12). Note that $Y_n \cap Y_{n+1} = \phi$ (empty) and $F_n \cap F_{n+1} = \phi$. Equation (12) can also be multiplied by A_1^{-1} and written in the form

$$Y_{n+1} = A_1^{-1}A_0Y_n + h(A_1^{-1}B_1F_{n+1} + A_1^{-1}B_0F_n) \quad (16)$$

where

$$A_1^{-1}A_0 = \left(\begin{array}{c|c} & 1 \\ & 1 \\ O & \cdot \\ & \cdot \\ & \cdot \\ & 1 \end{array}\right) = (O \quad e) ; \quad A_1^{-1}B_0 = \left(\begin{array}{c|c} & \times \\ & \times \\ O & \cdot \\ & \cdot \\ & \cdot \\ & \times \end{array}\right) \quad or \quad A_1^{-1}B_0 = O$$

$$A_1^{-1}B_1 = \begin{pmatrix} \times & \cdot & \cdot & \cdot & \times \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \times \end{pmatrix};$$

The structure of $A_1^{-1}A_0$ and $A_1^{-1}B_0$ makes the need to carry along the past solution $Y_n = (y_{n-2k+l+1}, y_{n-2k+l+2}, \ldots, y_{n-1}, y_n)^T$; unnecessary. Only the initial solution y_n provided by the ODE (1) is needed to implement the block method (12). Therefore (12) is self-starting block methods.

The coefficients of the order nine of the methods are given here below:

$$A_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}; \quad A_0 = \begin{pmatrix} O_{2\times15} & \begin{matrix} -1 \\ 0 \end{matrix} \\ O_{14\times15} & O_{14\times1} \end{pmatrix}; \quad B_0 = \begin{pmatrix} O_{2\times15} \\ O_{14\times15} \end{pmatrix}$$

$$B_1 \square \left[\begin{array}{ccccccccccccccc}
\frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} & \frac{8183}{1036800} & 0 & 0 & 0 & 0 & 0 & 0 \\
\square\frac{116687}{1451520} & \frac{335983}{907200} & \square\frac{462127}{453600} & \frac{6755041}{3628800} & \square\frac{8641823}{3628800} & \frac{200029}{90720} & \square\frac{1408913}{907200} & \frac{9449717}{7257600} & \frac{25713}{89600} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{25713}{89600} & \frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} & \frac{8183}{1036800} & 0 & 0 & 0 & 0 & 0 \\
\frac{8183}{1036800} & \square\frac{116687}{1451520} & \frac{335983}{907200} & \square\frac{462127}{453600} & \frac{6755041}{3628800} & \square\frac{8641823}{3628800} & \frac{200029}{90720} & \square\frac{1408913}{907200} & \frac{9449717}{7257600} & \frac{25713}{89600} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{25713}{89600} & \frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} & \frac{8183}{1036800} & 0 & 0 & 0 & 0 \\
0 & \frac{8183}{1036800} & \square\frac{116687}{1451520} & \frac{335983}{907200} & \square\frac{462127}{453600} & \frac{6755041}{3628800} & \square\frac{8641823}{3628800} & \frac{200029}{90720} & \square\frac{1408913}{907200} & \frac{9449717}{7257600} & \frac{25713}{89600} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{25713}{89600} & \frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} & \frac{8183}{1036800} & 0 & 0 & 0 \\
0 & 0 & \frac{8183}{1036800} & \square\frac{116687}{1451520} & \frac{335983}{907200} & \square\frac{462127}{453600} & \frac{6755041}{3628800} & \square\frac{8641823}{3628800} & \frac{200029}{90720} & \square\frac{1408913}{907200} & \frac{9449717}{7257600} & \frac{25713}{89600} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{25713}{89600} & \frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} & \frac{8183}{1036800} & 0 & 0 \\
0 & 0 & 0 & \frac{8183}{1036800} & \square\frac{116687}{1451520} & \frac{335983}{907200} & \square\frac{462127}{453600} & \frac{6755041}{3628800} & \square\frac{8641823}{3628800} & \frac{200029}{90720} & \square\frac{1408913}{907200} & \frac{9449717}{7257600} & \frac{25713}{89600} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{25713}{89600} & \frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} & \frac{8183}{1036800} & 0 \\
0 & 0 & 0 & 0 & \frac{8183}{1036800} & \square\frac{116687}{1451520} & \frac{335983}{907200} & \square\frac{462127}{453600} & \frac{6755041}{3628800} & \square\frac{8641823}{3628800} & \frac{200029}{90720} & \square\frac{1408913}{907200} & \frac{9449717}{7257600} & \frac{25713}{89600} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{25713}{89600} & \frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} & \frac{8183}{1036800} \\
0 & 0 & 0 & 0 & 0 & \frac{8183}{1036800} & \square\frac{116687}{1451520} & \frac{335983}{907200} & \square\frac{462127}{453600} & \frac{6755041}{3628800} & \square\frac{8641823}{3628800} & \frac{200029}{90720} & \square\frac{1408913}{907200} & \frac{9449717}{7257600} & \frac{25713}{89600} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{25713}{89600} & \frac{9449717}{7257600} & \square\frac{1408913}{907200} & \frac{200029}{90720} & \square\frac{8641823}{3628800} & \frac{6755041}{3628800} & \square\frac{462127}{453600} & \frac{335983}{907200} & \square\frac{116687}{1451520} \\
0 & 0 & 0 & 0 & 0 & 0 & 8183 & \square 116687 & 335983 & \square 462127 & 6755041 & \square 8641823 & 200029 & \square 1408913 & 9449717
\end{array}\right]$$

5. Stability of the Implicit Block Methods

When (12) is applied to the test equation

$$y' = \lambda y \, , \; \mathrm{Re}(\lambda) < 0 \quad (17)$$

it yields the characteristics equation.

$$\pi(w,z) = \det(A_1 w + A_0 - z(B_1 w + B_0)), \quad z = \lambda h \quad (18)$$

The region of absolute stability R_A associated with (12) is the set

$$R_A = \{z : \left|w_j(z)\right| \le 1, \; j = 1(1)k\} \quad (19)$$

If we let $z \to 0$ in (18), the difference system becomes

$$\pi(w,0) = \det(A_1 w + A_0) \qquad (20)$$

All the proposed block methods can be cast in the form

$$A_1 Y_{n+1} + \hat{a} y_n = h(B_1 F_{n+1} + \hat{b} f_n) \quad (21)$$

Where

$$\hat{a} = \begin{pmatrix} \alpha_0^{[1]} \\ \alpha_0^{[2]} \\ . \\ . \\ . \\ \alpha_0^{[m]} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}_{2(k-l)\times 1} ; \quad \hat{b} = \begin{pmatrix} \beta_0^{[1]} \\ \beta_0^{[2]} \\ . \\ . \\ . \\ \beta_0^{[m]} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}_{2(k-l)\times 1} \qquad (22)$$

International Journal of Scientific Engineering and Research (IJSER)
ISSN (Online): 2347-3878
Index Copernicus Value (2015): 62.86 | Impact Factor (2015): 3.791

Notice that for all the block methods, $A_1^{-1}\hat{a} = (1\,1\,1\ldots1)^T = e$

implying that

$$A_1^{-1}A_0 = \begin{pmatrix} O & \begin{matrix}1\\1\\ \cdot \\ \cdot \\ \cdot \\ 1\end{matrix} \end{pmatrix} = \begin{pmatrix} O & e \end{pmatrix}$$

To see this, assume order $p \geq 1$ for all the LMF that constitute the block, then by consistency,

$$A_1 e + \hat{a} = 0 \quad (23)$$

where $e = (1\,1\,1\ldots1)^T$. From (21) it follows that

$$A_1^{-1}\hat{a} = -e \quad (24)$$

The above ensures zero-stability of the implicit block methods (12).Method (12) can also be written as

$$Y_{n+1} = M(z)Y_n \quad (25)$$

where

$$M(z) = (I - zA_1^{-1}B_1)^{-1}(zA_1^{-1}B_0 - A_1^{-1}A_0) \quad (26)$$

is the amplification matrix. The stability function $p(w,z)$ is

$$p(w,z) = Det[I_k w - M(z)] = w^{k-1}(w - D(z)) \quad (27)$$

The stability domain S of this family is

$$S = \{z \in \mathrm{C} : |w(z)| \leq 1\} \quad (28)$$

The $D(z)$ (the only non-zero value of R(z)) for this family of methods are given as a rational function $D(z) = \dfrac{P(z)}{Q(z)}$.

where $P(z)$ and $Q(z)$ are polynomials.

Definition1: A block method is said to be pre-stable if the roots of $Q(z)$ are contained in C^+ (see [7]).

The one step block method is A-stable if and only if it is stable on the imaginary axis (I-stable): $D(iy) \leq 1$ for all $y \in \Re$, and D(z) is analytic for $D(z) < 0$ (i.e., $Q(z)$ does not have roots with negative or zero real parts), I-stability is equivalent to the fact that the Norsett polynomial defined by

$$E(y) = |Q(iy)|^2 - |P(iy)|^2 = Q(iy)Q(-iy) - P(iy)P(-iy) \quad (29)$$

satisfies $E(y) > 0$ for all $y \in \Re$ [13].

For this particular case oforder10, the characteristics equation $\pi(w,z)$ in (18) is given by

```
w14 □129443311994996077992345600000000000000000 w □129443311994996077992345600000000000000000 w2 □
 035546495959968623938764800000000000000000 w z □103554649595996862393876480000000000000000 w2 z □
 602404875855638895623078412288000000000000 w z2 □ 602404875855638895623078412288000000000000 w2
 261007314879717810058192905830400000000000 w z3 □261007314879717810058192905830400000000000 w
 799062697423561917929326345253191680000000 w z4 □799062697423561917929326345253191680000000 w
 176692296138830016967164668671033344000000 w z5 □176692296138830016967164668671033344000000
 292972493893544206535710083648456917222400 w z6 □29297249389354420653571008364845691722240
 375567494255980681948014276510762665779200 w z7 □37556749425598068194801427651076266577920
 380008358066428495850072782831891646592000 w z8 □38000835806642849585007278283189164659200
 307094375921890678763936410072184236646400 w z9 □30709437592189067876393641007218423664640
 199026272702679394972115970899395556794560 w z10 □1990262727026793949721159708993955567945
 103161683602319140827442857442233997002240 w z11 □1031616836023191408274428574422339970022
 423378976246588591082408544103266802272000 w z12 □423378976246588591082408544103266802272000
 134529769473328551243477322427210458675200 w z13 □134529769473328551243477322427210458675200
 314947681610968552687494546369013425100800 w z14 □314947681610968552687494546369013425100800 w2
 463047088792937170747062645021467086848 w z15 □463047088792937170747062645021467086848 w2 z
 708631861661104664939062633912303539530 w z16 □708631861661104664939062633912303539530 w2 z16
129443311994996077992345600000000000000000
```

The roots of this characteristic equation is

```
w □ 0, w □ 0, w □ 0, w □ 0, w □ 0, w □ 0, w □ 0, w □ 0, w □ 0,
 w □ 0, w □ 0, w □ 0, w □ 0, w □ 0, w □ 0, w □ 0,
 w □ 129443311994996077992345600000000000000000 □103554649595996862393876480000000000000000 z □
  602404875855638895623078412288000000000000 z2 □261007314879717810058192905830400000000000
  799062697423561917929326345253191680000000 z4 □17669229613883001696716466867103334400000
  292972493893544206535710083648456917222400 z6 □37556749425598068194801427651076266577
  380008358066428495850072782831891646592000 z8 □30709437592189067876393641007218423664
  199026272702679394972115970899395556794560 z10 □103161683602319140827442857442233997000
  423378976246588591082408544103266802272000 z12 □134529769473328551243477322427210458675
  314947681610968552687494546369013425100800 z14 □463047088792937170747062645021467086848
  708631861661104664939062633912303539530 z16 □129443311994996077992345600000000000000000 □
 103554649595996862393876480000000000000000 z □ 602404875855638895623078412288000000000000 z2 □
 261007314879717810058192905830400000000000 z3 □799062697423561917929326345253191680000000 z
 176692296138830016967164668671033344000000 z5 □29297249389354420653571008364845691722240
 375567494255980681948014276510762665779200 z7 □38000835806642849585007278283189164659200
 307094375921890678763936410072184236646400 z9 □19902627270267939497211597089939555679456
 103161683602319140827442857442233997002240 z11 □423378976246588591082408544103266802272000
 134529769473328551243477322427210458675200 z13 □314947681610968552687494546369013425100800 z
 463047088792937170747062645021467086848 z15 □708631861661104664939062633912303539530 z16
```

For this particular case, the only none zero solution $D(z)$, has no pole on C^{-}, all the roots of $Q(z)$ which is the denominator of the rational function $D(z)$ are contained in C^{+} as shown below

z = 9.3622e-12-0.2518i, z = 9.3622e-12+0.2518i, z = 0.0991 -1.8292i, z = 0.0991 +1.8292i,
z = 0.1453 -4.3834i, z = 0.1453 +4.3834i, z = 0.3053 -1.1253i, z = 0.3053 +1.1253i, z = 0.5462 - 0.843i, z = 0.5462+0.8433i, z = 0.6585 -0.6008i, z = 0.6585 + 0.6008i, z = 0.7343 - 0.3490i, z = 0.7343 + 0.3490i, z = 0.7786 -0.1134i,z = 0.7786 + 0.1134i

$E(y) > 0$ in(29) for all $y \in \Re$,the method is therefore A-Stable

6. Numerical Experiments

In this section, we considered two problems to test the effectiveness of the method

Problem 1: Singularly Perturbed Problem (cf: [12])

$$y_1'(t) = -(2+\varepsilon^{-1})y_1(t) + \varepsilon^{-1}y_2^2(t);\; y_1(0)=1$$
$$y_2'(t) = y_1(t) - y_2(t) - y_2^2(t);\; y_2(0)=1$$
$$t\in[0,1];\; \varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$$

The theoretical solutions are $y_1(t)=e^{-2t};\; y_2(t)=e^{-t}$. The errors when our method of order 9 is implemented on problem 1 are given in table 1. Even when the value of ε is tending to zero, the errors are consistent.

Table 1: errors on different vales of ε on problem 2 using RAM/AM k=9

h=0.0001	Errors
$\varepsilon = 10^{-4}$	3.58e-08
$\varepsilon = 10^{-3}$	6.29e-07
$\varepsilon = 10^{-2}$	4.05e-07
$\varepsilon = 10^{-1}$	5.01e-08

Problem 2:

Consider the following linear constant coefficient initial value problem taken from [7],

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y; \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The theoretical solution is given by

$$y(t) = \frac{1}{2}\begin{pmatrix} e^{-2t} + e^{-40t}(\cos(40t)+\sin(40t)) \\ e^{-2t} - e^{-40t}(\cos(40t)+\sin(40t)) \\ 2e^{-40t}(\sin(40t)-\cos(40t)) \end{pmatrix}$$

The errors and computational rate of convergence computed with our method of order 10 at different values of h are given in table 2

Table 2: errors and convergence rate of RAM/AM k=9 on problem 2

h	Errors	Convergence rate
0.032	1.70e-02	
0.016	2.06e-04	6.3650
0.008	1.02e-06	7.6567
0.004	3.61e-09	8.1409
0.002	4.55e-12	9.6330
0.001	7.35e-15	9.2754

7. Conclusion

The work in [1] has been extended to higher order methods. The order ten of the family is $A-Stable$ while the order eleven is $A(\alpha)-Stable$ with $\alpha = 74.5934^o$. The errors from the implementation of the method of order ten on some stiff initial value problems shows that they are effective.

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