

# \* Derivations in Prime Gamma Nearrings

Dr. D. Bharathi<sup>1</sup>, K. Sreenivasulu<sup>2</sup>

<sup>1</sup>Professor, Department of Mathematics, S.V.U, Tirupathi, A.P., India

<sup>2</sup>Research scholar, Department of Mathematics, S.V.U, Tirupathi, A.P., India  
ksreenugke [at]gmail.com

**Abstract:** Let  $N$  be a prime  $\Gamma$ -nearring of a 2-torsion free and let  $D_1$  and  $D_2$  be a  $*$  derivations with the condition  $D_1(a)\alpha D_2(b)=D_2(b)\alpha D_1(a)$  for all  $a, b \in N$  and  $\alpha \in \Gamma$  on  $N$ . Then  $D_1 D_2$  is a  $*$  derivation on  $N$  if and only if either  $D_1=0$  or  $D_2=0$ .

**Keywords:**  $*$  derivation,  $\Gamma$ -nearring, 2-torsion free

## 1. Introduction

The notion of derivations in nearrings introduced by Bell [1] and Mason. They obtained some basic properties of derivations in nearrings and some commutativity conditions for a  $\Gamma$ - $*$  nearrings with derivations. Cho [2] studied some characterization of  $\Gamma$ - $*$  nearrings and some regularity conditions.

In classical ring theory, Posner [3], Herstein [4], Bergen [5], Bell and Daif [6] worked on derivations in prime and semiprime rings and gets some commutativity properties of prime rings with derivations. In nearring theory, Bell and Mason [1] and also Cho [7] worked on derivations in prime and semiprime nearrings.

This paper, we slightly extended the results of Cho [7] in prime  $\Gamma$ - $*$  nearrings with certain conditions by using derivations.

## 2. Main Theorems

**Lemma 1:** Let  $D$  be an arbitrary additive endomorphism of  $N$ . Then  $D(\alpha\beta)=\alpha D(\beta)+D(\alpha)\beta^*$  if and only if  $D(\alpha\beta)=D(\alpha)\beta^*+\alpha D(\beta)$  for all  $a, b \in N$  and  $\alpha \in \Gamma$ .

**Proof:** Assume that  $D(\alpha\beta)=D(\alpha)\beta^*+\alpha D(\beta)$  for all  $a, b \in N$  and  $\alpha \in \Gamma$ . For  $\alpha \in \Gamma$  and from

$$\alpha(b+b)=\alpha b+\alpha b$$

and  $N$  satisfies left distributive law,

$$D(\alpha(b+b))=\alpha D(b+b)+D(\alpha)(b^*+b^*)$$

$$= \alpha(D(b)+D(b))+D(\alpha)\alpha b^*+D(\alpha)\alpha b^*$$

$$= \alpha D(b)+\alpha D(b)+D(\alpha)\alpha b^*+D(\alpha)\alpha b^*$$

and  $D(\alpha\beta+\alpha\beta)=D(\alpha\beta)+D(\alpha\beta)$

$$=D(\alpha)\alpha b^*+\alpha D(b)+D(\alpha)\alpha b^*+\alpha D(b)$$

By comparing these two equations, we have

$$\alpha D(b)+D(\alpha)\alpha b^*=D(\alpha)\alpha b^*+\alpha D(b)$$

Hence  $D(\alpha\beta)=D(\alpha)\alpha b^*+\alpha D(b)$  for all  $a, b \in N$  and  $\alpha \in \Gamma$ .

Conversely, assume that  $D(\alpha\beta)=D(\alpha)\alpha b^*+\alpha D(b)$  for all  $a, b \in N$  and  $\alpha \in \Gamma$

Then form  $D(\alpha(b+b))=D(\alpha\beta+\alpha\beta)$  and we can induce that, the above calculate of this equality  $D(\alpha\beta)=\alpha D(b)+D(\alpha)\alpha b^*$  for all  $a, b \in N$  and  $\alpha \in \Gamma$ .  $\diamond$

**Lemma 2:** Let  $D$  be a  $*$  derivation on  $N$ . Then  $N$  satisfies the following right distributive laws,  $a, b, c \in N$  and  $\alpha, \beta \in \Gamma$ .

$$\{\alpha D(b)+D(\alpha)\alpha b^*\}\beta c = \alpha D(b)\beta c + D(\alpha)\alpha b^*\beta c,$$

$$\{D(a)\alpha b^*+\alpha D(b)\}\beta c = D(a)\alpha b^*\beta c + \alpha D(b)\beta c$$

**Proof:** From the calculation for  $D((\alpha\beta)\beta c)=D(\alpha(\beta\beta c))$  for all  $a, b, c \in N$  and  $\alpha, \beta \in \Gamma$  and Lemma 1, we can find our result.

**Lemma 3:** Let  $N$  be a prime  $\Gamma$ -nearring and  $U$  be a nonzero subset of  $N$ . If 'a' be an element of  $N$  such that  $U\Gamma a=\{0\}$ , then  $a=0$ .

**Proof:** Since  $U \neq \{0\}$ , there exists an element  $u \in U$  such that  $u \neq 0$ . Consider that  $u\Gamma N\Gamma a \subset U\Gamma a = \{0\}$ . Since  $u \neq 0$  and  $N$  is a prime  $\Gamma$ -nearring, we have that  $a=0$ .  $\diamond$

**Lemma 4:** Let  $N$  be a prime  $\Gamma$ -nearring and  $U$  a nonzero  $N$ -subset of  $N$ . If  $D$  is a nonzero  $*$  derivation on  $N$  then

(i) If  $a, b \in N$  and  $a\Gamma U\Gamma b = \{0\}$ , then  $a=0$  or  $b=0$

(ii) If  $a \in N$  and  $D(U)\Gamma a = \{0\}$ , then  $a=0$

(iii) If  $a \in N$  and  $a\Gamma D(U) = \{0\}$ , then  $a=0$ .

**Proof:** (i) Let  $a, b \in N$  and  $a\Gamma U\Gamma b = \{0\}$

Then  $a\Gamma U\Gamma N\Gamma b \subset a\Gamma U\Gamma b = \{0\}$ .

Since  $N$  is a prime  $\Gamma$ -nearring,  $a\Gamma U=0$  or  $b=0$

If  $b=0$ , then we are done.

So, if  $b \neq 0$ , then  $a\Gamma U=0$

By the lemma 3, then  $a=0$

(ii) Suppose  $D(U)\Gamma a = \{0\}$ , for  $a \in N$ . Then for all  $u \in U$  and  $b \in N$ , by the lemma, we have

$$\text{for all } a, b \in N \text{ and } \alpha, \beta \in \Gamma, 0 = D(b\alpha u)\beta a = (b\alpha D(u) + D(b)\alpha u^*)\beta a$$

$$= b\alpha D(u)\beta a + D(b)\alpha u^*\beta a$$

So  $D(b)\Gamma U\Gamma a = \{0\}$ , for all  $b \in N$ . (Since  $D(U)\Gamma a = \{0\}$ )

Since  $D$  is a nonzero  $*$  derivation on  $N$ , we have that  $a=0$  by the statement (i)

(iii) Suppose  $a\Gamma D(U) = \{0\}$  for  $a \in N$ . Then for all  $u \in U$ ,  $b \in N$  and  $\alpha, \beta \in \Gamma$ .

$$0 = \alpha D(u\beta b) = \alpha \{ u\beta D(b) + D(u)\beta b^* \}$$

$$= \alpha u\beta D(b) + \alpha D(u)\beta b^*$$

$$= \alpha u\beta D(b)$$

Hence  $a\Gamma U\Gamma D(b) = \{0\}$  for all  $b \in N$  from the statement (i) and  $D$  is a nonzero  $*$  derivation on  $N$ , we have that  $a=0$

We assume that  $N$  is a  $\Gamma$ -ring and  $U$  is a right  $N$ -subset.  $\diamond$

**Theorem 1:** Let  $N$  be a prime  $\Gamma$ -nearring and  $U$  be a right  $N$ -subset of  $N$ . If  $D$  is a nonzero  $*$  derivation on  $N$  such that  $D^2(u)=0$  then  $D^2=0$

**Proof:** For all  $u, v \in U$  and  $\alpha \in \Gamma$ ,

We have  $D^2(u\alpha v)=0$ .

Volume 5 Issue 7, July 2017

[www.ijser.in](http://www.ijser.in)

Licensed Under Creative Commons Attribution CC BY

Then  $0 = D^2(u\alpha v) = D(D(u\alpha v)) = D\{D(u)\alpha v^* + u\alpha D(v)\}$   
 $= D^2(u)\alpha(v^*)^* + D(u)\alpha D(v^*) + D(u)\alpha D(v^*) + u\alpha D^2(v)$   
 Thus  $D(U)\Gamma D(U) = \{0\}$  for all  $u \in U$ .

From lemma 4(iii), we have  $D(U) = 0$ .

Now, for all  $b \in N, u \in U$  and  $\alpha \in \Gamma$

$$D^2(u\alpha b) = D(D(u\alpha b)) = D(D(u)\alpha b^* + u\alpha D(b))$$

$$= u\alpha D^2(b) + D^2(u)\alpha b^* + D(u)\alpha D(b) + D(u)\alpha D(b^*)$$

Hence  $U\Gamma D^2(b) = \{0\}$  for all  $b \in N$ .

By the lemma 4, we have  $D^2(b) = 0$  for all  $b \in N$ . Consequently  $D^2 = 0$ .  $\diamond$

**Lemma 4:** Let  $D$  be a  $*$  derivation of a prime  $\Gamma$ -nearing  $N$  and 'a' be an element of  $N$ . If  $\alpha a D(x) = 0$  (or  $D(x)\alpha a = 0$ ) for all  $x \in N, \alpha \in \Gamma$ , then either  $a = 0$  or  $D = 0$ .

**Proof:** We assume that  $\alpha a D(x) = 0$  for all  $x \in N, \alpha \in \Gamma$ .

Change  $x$  by  $x\beta y$  (for all  $\beta \in \Gamma$ ), we have that

$$\alpha a D(x\beta y) = 0 = \alpha a D(x) \beta y^* + \alpha a x \beta D(y).$$

Then by lemma 2,

Since  $\alpha a D(x) = 0, \alpha a x \beta D(y) = 0$  for all  $x, y \in N, \alpha, \beta \in \Gamma$ .

Since  $N$  is a  $\Gamma$ -prime nearing, either  $a = 0$  or  $D = 0$ .  $\diamond$

**Theorem 2:** Let  $N$  be a  $\Gamma$ -prime nearing with nonzero  $*$  derivation and  $D_1$  and  $D_2$  such that for all  $x, y \in N$  and  $\alpha \in \Gamma, D_1(x)\alpha D_2(y) = -D_2(x)\alpha D_1(y)$ . Then  $N$  is an abelian  $\Gamma$ -nearing

**Proof:** Let  $x, u, v \in N, \alpha \in \Gamma$ .

$$\text{We have } D_1(x)\alpha D_2(y) = -D_2(x)\alpha D_1(y)$$

Substitute  $y$  by  $u+v$

$$0 = D_1(x)\alpha D_2(u+v) + D_2(x)\alpha [D_1(u) + D_1(v)]$$

$$= D_1(x)\alpha [D_2(u) + D_2(v)] + D_2(x)\alpha [D_1(u) + D_1(v)]$$

$$= D_1(x)\alpha D_2(u) + D_1(x)\alpha D_2(v) - D_1(x)\alpha D_2(u) - D_1(x)\alpha D_2(v)$$

$$= D_1(x)\alpha [D_2(u) + D_2(v) - D_2(u) - D_2(v)]$$

$$= D_1(x)\alpha D_2(u+v-u-v)$$

$$\text{Thus } D_1(N)\Gamma D_2(u+v-u-v) = \{0\} \quad (1)$$

By the Lemma 4, we have

$$D_2(u+v-u-v) = \{0\} \quad (2)$$

Now, we substitute  $x\beta u$  and  $x\beta v$  ( $\beta \in \Gamma$ ) instead of  $u$  and  $v$  respectively in (2). Then from Lemma 1, we deduce that for all  $x, u, v \in N, \beta \in \Gamma$ .

$$0 = D_2(x\beta u + x\beta v - x\beta u - x\beta v)$$

$$= D_2[x\beta(u+v-u-v)]$$

$$= D_2(x)\beta(u+v-u-v)^* + x\beta D_2(u+v-u-v) \quad (\text{by Lemma 4})$$

$$= D_2(x)\beta(u+v-u-v)^*$$

Substitute  $(u+v-u-v)^*$  by  $u+v-u-v$ .

Again apply by Lemma 4, we see that for all  $u, v \in N, u+v-u-v = 0$ .

$$\Rightarrow u+v = v+u$$

Consequently,  $N$  is an abelian  $\Gamma$ -nearing.  $\diamond$

**Theorem 3:** Let  $N$  be a prime  $\Gamma$ -nearing of 2-torsion free and let  $D_1$  and  $D_2$  be  $*$  derivations with the condition  $D_1(a)\alpha D_2(b) = D_2(b)\alpha D_1(a)$  for all  $a, b \in N, \alpha \in \Gamma$  on  $N$ . Then  $D_1 D_2$  is a  $*$  derivation on  $N$  if and only if either  $D_1 = 0, D_2 = 0$ .

**Proof:** We assume that  $D_1 D_2$  is a  $*$  derivation. Then we get for  $\alpha \in \Gamma$ ,

$$D_1 D_2(a\alpha b) = \alpha a D_1 D_2(b) + D_1 D_2(a)\alpha b^* \quad (3)$$

Also, since  $D_1$  and  $D_2$  are  $*$  derivations, we obtain

$$D_1 D_2(a\alpha b) = D_1(D_2(a\alpha b)) = D_1(\alpha a D_2(b) + D_2(a)\alpha b^*)$$

$$= D_1(\alpha a D_2(b)) + D_1(D_2(a)\alpha b^*)$$

$$= \alpha a D_1 D_2(b) + D_1(a)\alpha D_2(b^*) + D_2(a)\alpha D_1(b)^* + D_1 D_2(a)\alpha b^*$$

$$= \alpha a D_1 D_2(b) + D_1(a)\alpha D_2(b^*) + D_2(a)\alpha D_1(b) + D_1 D_2(a)\alpha b^* \quad (4)$$

From (3) and (4) for  $D_1 D_2(a\alpha b)$  for all  $a, b \in N, \alpha \in \Gamma$ ,

$$D_1(a)\alpha D_2(b^*) + D_2(a)\alpha D_1(b) = 0 \quad (5)$$

Hence from Theorem 2, we know that  $N$  is an abelian  $\Gamma$ -nearing.

Substitute  $a$  by  $\alpha a D_2(c)$  in (5) and using Lemma 1 and  $*$  derivations in right distributive laws, we get that

$$0 = D_1(\alpha a D_2(c))\alpha D_2(b^*) + D_2(\alpha a D_2(c))\alpha D_1(b)$$

$$= \{D_1(a)\alpha D_2(c^*) + \alpha a D_1 D_2(c)\}\alpha D_2(b^*) + \{ \alpha a D_2^2(c^*) + D_2(a)\alpha D_2(c)\}\alpha D_1(b)$$

$$= D_1(a)\alpha D_2(c^*) \alpha D_2(b^*) + \alpha a D_1 D_2(c)\alpha D_2(b^*) + \alpha a D_2^2(c^*) \alpha D_1(b) + D_2(a)\alpha D_2(c)\alpha D_1(b)$$

$$= D_1(a)\alpha D_2(c^*) \alpha D_2(b^*) + \alpha a \{D_1 D_2(c)\alpha D_2(b^*) + D_2^2(c^*) \alpha D_1(b)\} + D_2(a)\alpha D_2(c)\alpha D_1(b)$$

Again change  $a$  by  $D_2(c)$  in (5), we see that

$$D_1(D_2(c)\alpha D_2(b^*)) + D_2(D_2(c))\alpha D_1(b) = 0$$

This implies that  $\alpha a \{D_1 D_2(c)\alpha D_2(b^*) + D_2^2(c)\alpha D_1(b)\} = 0$

Hence, from the above long equality, we have the following equality,

$$D_1(a)\alpha D_2(c^*) \alpha D_2(b^*) + D_2(a)\alpha D_2(c) \alpha D_1(b) = 0, \text{ for all } a, b \in N, \alpha \in \Gamma. \quad (6)$$

Replace  $a$  and  $b$  by  $c$  in (5), we get

$$D_2(c) \alpha D_1(b) = -D_1(c) \alpha D_2(b^*), D_1(a)\alpha D_2(c^*) = -D_2(a) \alpha D_1(c)$$

So that (6) becomes,

$$0 = \{-D_2(a) \alpha D_1(c^*)\} \alpha D_2(b^*) + D_2(a)\alpha \{-D_1(c) \alpha D_2(b^*)\}$$

$$= D_2(a) \alpha (-D_1(c^*)) \alpha D_2(b^*) + D_2(a)\alpha (-D_1(c))\alpha D_2(b^*)$$

$$= D_2(a)\beta \{-D_1(c^*)\} \alpha D_2(b^*) - D_1(c)\alpha D_2(b^*), \text{ for all } a, b \in N, \alpha \in \Gamma.$$

If  $D_2 \neq 0$  then by Lemma 4,

We have the equality,  $(-D_1(c^*))\alpha D_2(b^*) - D_1(c)\alpha D_2(b^*) = 0$ .

That is  $D_1(c^*)\alpha D_2(b^*) = (-D_1(c))\alpha D_2(b^*)$ , for all  $b, c \in N, \alpha \in \Gamma$ . (7)

Thus, using this condition of our theorem, we obtain

$$(-D_1(c))\alpha D_2(b^*) = D_1(-c)\alpha D_2(b^*) = -D_2(b^*)\alpha D_1(-c) = -D_2(b^*)\alpha (-D_1(c)) = -D_2(b^*)\alpha D_1(c)$$

$$= -D_1(c)\alpha D_2(b^*) \quad (8)$$

From (7) and (8) we have that, for all  $b, c \in N, \alpha \in \Gamma$ ,

$$2D_1(c)\alpha D_2(b^*) = 0.$$

Since  $N$  is 2-torsion free,  $D_1(c)\alpha D_2(b^*) = 0$

Also, since  $D_2$  is not zero, by Lemma 4, we see that  $D_1(c) = 0$  for all  $c \in N$ . Therefore  $D_1 = 0$

Consequently, either  $D_1 = 0$  or  $D_2 = 0$ .

The converse verification is obvious  $\diamond$ .

## References

- [1] Bell H.E. and Mason G., On derivations in nearings, In: Gerhard Betsch(Ed), Near-Rings and Near-Fields, Proceedings of the conference held at the University of Tubingen, Tubingen, August 4-10, 1985(pp.31-35).Noth-Holland, Amsterdam, 1987.
- [2] Cho, Y.U., A study on derivations in near-rings. Pusan Kyongnam Math. J., 12(1) (1996), 63-69.
- [3] Posner, E.C., Derivations in prime rings. Proc. Amer. Math. Soc.8 (1957), 1093-1100.
- [4] Herstein I.N., A note on derivations. Canad. Math. Bull. 21(3)(1978), 369-370.
- [5] Bergen J., Derivations in prime rings. Canad. Math. Bull. 26(1983), no.3, 267-277.
- [6] Bell H.E. and Daif M. N, On derivations and commutativity in prime rings, Acta. Math.Hungar. 66(4)(1995), 337-343.
- [7] Yong Uk Cho, Some conditions on derivations in prime near rings, J Korea Soc Math Educ Ser B Pure Appl Math 8(2) 2001, 145-152