

* Derivations in Prime Gamma Nearrings

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Abstract: Let N be a prime Γ -nearring of a 2-torsion free and let D_1 and D_2 be a $*$ derivations with the condition $D_1(a)\alpha D_2(b)=D_2(b)\alpha D_1(a)$ for all $a, b \in N$ and $\alpha \in \Gamma$ on N . Then $D_1 D_2$ is a $*$ derivation on N if and only if either $D_1=0$ or $D_2=0$.

Keywords: $*$ derivation, Γ -nearring, 2-torsion free

1. Introduction

The notion of derivations in nearrings introduced by Bell [1] and Mason. They obtained some basic properties of derivations in nearrings and some commutativity conditions for a Γ - $*$ nearrings with derivations. Cho [2] studied some characterization of Γ - $*$ nearrings and some regularity conditions.

In classical ring theory, Posner [3], Herstein [4], Bergen [5], Bell and Daif [6] worked on derivations in prime and semiprime rings and gets some commutativity properties of prime rings with derivations. In nearring theory, Bell and Mason [1] and also Cho [7] worked on derivations in prime and semiprime nearrings.

This paper, we slightly extended the results of Cho [7] in prime Γ - $*$ nearrings with certain conditions by using derivations.

2. Main Theorems

Lemma 1: Let D be an arbitrary additive endomorphism of N . Then $D(\alpha\beta)=\alpha D(\beta)+D(\alpha)\beta^*$ if and only if $D(\alpha\beta)=D(\alpha)\beta^*+\alpha D(\beta)$ for all $a, b \in N$ and $\alpha \in \Gamma$.

Proof: Assume that $D(\alpha\beta)=D(\alpha)\beta^*+\alpha D(\beta)$ for all $a, b \in N$ and $\alpha \in \Gamma$. For $\alpha \in \Gamma$ and from

$$\alpha(b+b)=\alpha b+\alpha b$$

and N satisfies left distributive law,

$$D(\alpha(b+b))=\alpha D(b+b)+D(\alpha)(b^*+b^*)$$

$$= \alpha(D(b)+D(b))+D(\alpha)\beta^*+D(\alpha)\beta^*$$

$$= \alpha D(b)+\alpha D(b)+D(\alpha)\beta^*+D(\alpha)\beta^*$$

and $D(\alpha\beta+\alpha\beta)=D(\alpha\beta)+D(\alpha\beta)$

$$=D(\alpha)\beta^*+\alpha D(b)+D(\alpha)\beta^*+\alpha D(b)$$

By comparing these two equations, we have

$$\alpha D(b)+D(\alpha)\beta^*=D(\alpha)\beta^*+\alpha D(b)$$

Hence $D(\alpha\beta)=D(\alpha)\beta^*+\alpha D(b)$ for all $a, b \in N$ and $\alpha \in \Gamma$.

Conversely, assume that $D(\alpha\beta)=D(\alpha)\beta^*+\alpha D(b)$ for all $a, b \in N$ and $\alpha \in \Gamma$

Then form $D(\alpha(b+b))=D(\alpha\beta+\alpha\beta)$ and we can induce that, the above calculate of this equality $D(\alpha\beta)=\alpha D(b)+D(\alpha)\beta^*$ for all $a, b \in N$ and $\alpha \in \Gamma$. \diamond

Lemma 2: Let D be a $*$ derivation on N . Then N satisfies the following right distributive laws, $a, b, c \in N$ and $\alpha, \beta \in \Gamma$.

$$\{\alpha D(b)+D(\alpha)\beta^*\}\beta c = \alpha D(b)\beta c + D(\alpha)\beta^*\beta c,$$

$$\{D(a)\alpha\beta^*+\alpha D(b)\}\beta c = D(a)\alpha\beta^*\beta c + \alpha D(b)\beta c$$

Proof: From the calculation for $D((\alpha\beta)\beta c)=D(\alpha(\beta\beta c))$ for all $a, b, c \in N$ and $\alpha, \beta \in \Gamma$ and Lemma 1, we can find our result.

Lemma 3: Let N be a prime Γ -nearring and U be a nonzero subset of N . If 'a' be an element of N such that $U\Gamma a=\{0\}$, then $a=0$.

Proof: Since $U \neq \{0\}$, there exists an element $u \in U$ such that $u \neq 0$. Consider that $u\Gamma N\Gamma a \subset U\Gamma a = \{0\}$. Since $u \neq 0$ and N is a prime Γ -nearring, we have that $a=0$. \diamond

Lemma 4: Let N be a prime Γ -nearring and U a nonzero N -subset of N . If D is a nonzero $*$ derivation on N then

(i) If $a, b \in N$ and $a\Gamma U\Gamma b = \{0\}$, then $a=0$ or $b=0$

(ii) If $a \in N$ and $D(U)\Gamma a = \{0\}$, then $a=0$

(iii) If $a \in N$ and $a\Gamma D(U) = \{0\}$, then $a=0$.

Proof: (i) Let $a, b \in N$ and $a\Gamma U\Gamma b = \{0\}$

Then $a\Gamma U\Gamma N\Gamma b \subset a\Gamma U\Gamma b = \{0\}$.

Since N is a prime Γ -nearring, $a\Gamma U = 0$ or $b=0$

If $b=0$, then we are done.

So, if $b \neq 0$, then $a\Gamma U = 0$

By the lemma 3, then $a=0$

(ii) Suppose $D(U)\Gamma a = \{0\}$, for $a \in N$. Then for all $u \in U$ and $b \in N$, by the lemma, we have

$$\text{for all } a, b \in N \text{ and } \alpha, \beta \in \Gamma, 0 = D(b\alpha u)\beta a = (b\alpha D(u) + D(b)\alpha u^*)\beta a$$

$$= b\alpha D(u)\beta a + D(b)\alpha u^*\beta a$$

So $D(b)\Gamma U\Gamma a = \{0\}$, for all $b \in N$. (Since $D(U)\Gamma a = \{0\}$)

Since D is a nonzero $*$ derivation on N , we have that $a=0$ by the statement (i)

(iii) Suppose $a\Gamma D(U) = \{0\}$ for $a \in N$. Then for all $u \in U$, $b \in N$ and $\alpha, \beta \in \Gamma$.

$$0 = \alpha D(u\beta b) = \alpha\{u\beta D(b) + D(u)\beta b^*\}$$

$$= \alpha u\beta D(b) + \alpha D(u)\beta b^*$$

$$= \alpha u\beta D(b)$$

Hence $a\Gamma U\Gamma D(b) = \{0\}$ for all $b \in N$ from the statement (i) and D is a nonzero $*$ derivation on N , we have that $a=0$

We assume that N is a Γ -ring and U is a right N -subset. \diamond

Theorem 1: Let N be a prime Γ -nearring and U be a right N -subset of N . If D is a nonzero $*$ derivation on N such that $D^2(u)=0$ then $D^2=0$

Proof: For all $u, v \in U$ and $\alpha \in \Gamma$,

We have $D^2(u\alpha v)=0$.

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Then $0 = D^2(u\alpha v) = D(D(u\alpha v)) = D\{D(u)\alpha v^* + u\alpha D(v)\}$
 $= D^2(u)\alpha(v^*)^* + D(u)\alpha D(v^*) + D(u)\alpha D(v^*) + u\alpha D^2(v)$

Thus $D(U)\Gamma D(U) = \{0\}$ for all $u \in U$.

From lemma 4(iii), we have $D(U) = 0$.

Now, for all $b \in N, u \in U$ and $\alpha \in \Gamma$

$$D^2(u\alpha b) = D(D(u\alpha b)) = D(D(u)\alpha b^* + u\alpha D(b))$$

$$= u\alpha D^2(b) + D^2(u)\alpha b^* + D(u)\alpha D(b) + D(u)\alpha D(b^*)$$

Hence $U\Gamma D^2(b) = \{0\}$ for all $b \in N$.

By the lemma 4, we have $D^2(b) = 0$ for all $b \in N$. Consequently $D^2 = 0$. \diamond

Lemma 4: Let D be a $*$ derivation of a prime Γ -nearing N and 'a' be an element of N . If $\alpha a D(x) = 0$ (or $D(x)\alpha a = 0$) for all $x \in N, \alpha \in \Gamma$, then either $a = 0$ or $D = 0$.

Proof: We assume that $\alpha a D(x) = 0$ for all $x \in N, \alpha \in \Gamma$.

Change x by $x\beta y$ (for all $\beta \in \Gamma$), we have that

$$\alpha a D(x\beta y) = 0 = \alpha a D(x) \beta y^* + \alpha a x \beta D(y).$$

Then by lemma 2,

Since $\alpha a D(x) = 0, \alpha a x \beta D(y) = 0$ for all $x, y \in N, \alpha, \beta \in \Gamma$.

Since N is a Γ -prime nearing, either $a = 0$ or $D = 0$. \diamond

Theorem 2: Let N be a Γ -prime nearing with nonzero $*$ derivation and D_1 and D_2 such that for all $x, y \in N$ and $\alpha \in \Gamma, D_1(x)\alpha D_2(y) = -D_2(x)\alpha D_1(y)$. Then N is an abelian Γ -nearing

Proof: Let $x, u, v \in N, \alpha \in \Gamma$.

$$\text{We have } D_1(x)\alpha D_2(y) = -D_2(x)\alpha D_1(y)$$

Substitute y by $u+v$

$$0 = D_1(x)\alpha D_2(u+v) + D_2(x)\alpha [D_1(u) + D_1(v)]$$

$$= D_1(x)\alpha [D_2(u) + D_2(v)] + D_2(x)\alpha [D_1(u) + D_1(v)]$$

$$= D_1(x)\alpha D_2(u) + D_1(x)\alpha D_2(v) - D_1(x)\alpha D_2(u) - D_1(x)\alpha D_2(v)$$

$$= D_1(x)\alpha [D_2(u) + D_2(v) - D_2(u) - D_2(v)]$$

$$= D_1(x)\alpha D_2(u+v-u-v)$$

$$\text{Thus } D_1(N)\Gamma D_2(u+v-u-v) = \{0\} \quad (1)$$

By the Lemma 4, we have

$$D_2(u+v-u-v) = \{0\} \quad (2)$$

Now, we substitute $x\beta u$ and $x\beta v$ ($\beta \in \Gamma$) instead of u and v respectively in (2). Then from Lemma 1, we deduce that for all $x, u, v \in N, \beta \in \Gamma$.

$$0 = D_2(x\beta u + x\beta v - x\beta u - x\beta v)$$

$$= D_2[x\beta(u+v-u-v)]$$

$$= D_2(x)\beta(u+v-u-v)^* + x\beta D_2(u+v-u-v) \text{ (by Lemma 4)}$$

$$= D_2(x)\beta(u+v-u-v)^*$$

Substitute $(u+v-u-v)^*$ by $u+v-u-v$.

Again apply by Lemma 4, we see that for all $u, v \in N, u+v-u-v = 0$.

$$\Rightarrow u+v = v+u$$

Consequently, N is an abelian Γ -nearing. \diamond

Theorem 3: Let N be a prime Γ -nearing of 2-torsion free and let D_1 and D_2 be $*$ derivations with the condition $D_1(a)\alpha D_2(b) = D_2(b)\alpha D_1(a)$ for all $a, b \in N, \alpha \in \Gamma$ on N . Then $D_1 D_2$ is a $*$ derivation on N if and only if either $D_1 = 0, D_2 = 0$.

Proof: We assume that $D_1 D_2$ is a $*$ derivation. Then we get for $\alpha \in \Gamma,$

$$D_1 D_2(a\alpha b) = \alpha a D_1 D_2(b) + D_1 D_2(a)\alpha b^* \quad (3)$$

Also, since D_1 and D_2 are $*$ derivations, we obtain

$$D_1 D_2(a\alpha b) = D_1(D_2(a\alpha b)) = D_1(\alpha a D_2(b) + D_2(a)\alpha b^*)$$

$$= D_1(\alpha a D_2(b)) + D_1(D_2(a)\alpha b^*)$$

$$= \alpha a D_1 D_2(b) + D_1(a)\alpha D_2(b^*) + D_2(a)\alpha D_1(b)^* + D_1 D_2(a)\alpha b^*$$

$$= \alpha a D_1 D_2(b) + D_1(a)\alpha D_2(b^*) + D_2(a)\alpha D_1(b) + D_1 D_2(a)\alpha b^* \quad (4)$$

From (3) and (4) for $D_1 D_2(a\alpha b)$ for all $a, b \in N, \alpha \in \Gamma,$

$$D_1(a)\alpha D_2(b^*) + D_2(a)\alpha D_1(b) = 0 \quad (5)$$

Hence from Theorem 2, we know that N is an abelian Γ -nearing.

Substitute a by $\alpha a D_2(c)$ in (5) and using Lemma 1 and $*$ derivations in right distributive laws, we get that

$$0 = D_1(\alpha a D_2(c))\alpha D_2(b^*) + D_2(\alpha a D_2(c))\alpha D_1(b)$$

$$= \{D_1(a)\alpha D_2(c^*) + \alpha a D_1 D_2(c)\}\alpha D_2(b^*) + \{ \alpha a D_2^2(c^*) + D_2(a)\alpha D_2(c)\}\alpha D_1(b)$$

$$= D_1(a)\alpha D_2(c^*) \alpha D_2(b^*) + \alpha a D_1 D_2(c)\alpha D_2(b^*) + \alpha a D_2^2(c^*) \alpha D_1(b) + D_2(a)\alpha D_2(c)\alpha D_1(b)$$

$$= D_1(a)\alpha D_2(c^*) \alpha D_2(b^*) + \alpha a \{D_1 D_2(c)\alpha D_2(b^*) + D_2^2(c^*) \alpha D_1(b)\} + D_2(a)\alpha D_2(c)\alpha D_1(b)$$

Again change a by $D_2(c)$ in (5), we see that

$$D_1(D_2(c)\alpha D_2(b^*)) + D_2(D_2(c))\alpha D_1(b) = 0$$

This implies that $\alpha a \{D_1 D_2(c)\alpha D_2(b^*) + D_2^2(c)\alpha D_1(b)\} = 0$

Hence, from the above long equality, we have the following equality,

$$D_1(a)\alpha D_2(c^*) \alpha D_2(b^*) + D_2(a)\alpha D_2(c) \alpha D_1(b) = 0, \text{ for all } a, b \in N, \alpha \in \Gamma. \quad (6)$$

Replace a and b by c in (5), we get

$$D_2(c) \alpha D_1(b) = -D_1(c) \alpha D_2(b^*), D_1(a)\alpha D_2(c^*) = -D_2(a) \alpha D_1(c)$$

So that (6) becomes,

$$0 = \{-D_2(a) \alpha D_1(c^*)\} \alpha D_2(b^*) + D_2(a)\alpha \{-D_1(c) \alpha D_2(b^*)\}$$

$$= D_2(a) \alpha \{-D_1(c^*)\} \alpha D_2(b^*) + D_2(a)\alpha \{-D_1(c)\} \alpha D_2(b^*)$$

$$= D_2(a)\beta \{-D_1(c^*)\} \alpha D_2(b^*) - D_1(c)\alpha D_2(b^*), \text{ for all } a, b \in N, \alpha \in \Gamma.$$

If $D_2 \neq 0$ then by Lemma 4,

We have the equality, $(-D_1(c^*))\alpha D_2(b^*) - D_1(c)\alpha D_2(b^*) = 0$.

That is $D_1(c^*)\alpha D_2(b^*) = (-D_1(c))\alpha D_2(b^*)$, for all $b, c \in N, \alpha \in \Gamma$. (7)

Thus, using this condition of our theorem, we obtain

$$(-D_1(c))\alpha D_2(b^*) = D_1(-c)\alpha D_2(b^*) = -D_2(b^*)\alpha D_1(-c) = D_2(b^*)\alpha (-D_1(c)) = -D_2(b^*)\alpha D_1(c)$$

$$= -D_1(c)\alpha D_2(b^*) \quad (8)$$

From (7) and (8) we have that, for all $b, c \in N, \alpha \in \Gamma,$

$$2D_1(c)\alpha D_2(b^*) = 0.$$

Since N is 2-torsion free, $D_1(c)\alpha D_2(b^*) = 0$

Also, since D_2 is not zero, by Lemma 4, we see that $D_1(c) = 0$ for all $c \in N$. Therefore $D_1 = 0$

Consequently, either $D_1 = 0$ or $D_2 = 0$.

The converse verification is obvious \diamond .

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