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* Derivations in Prime Gamma Nearrings

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Abstract: ***Let N be a prime Γ-nearring of a 2-torsion free and let D_1 and D_2 be a * derivations with the condition $D_1(a)\alpha D_2(b)=D_2(b)\alpha D_1(a)$ for all a, bϵN and $\alpha\epsilon\Gamma$ on N. Then D_1D_2 is a * derivation on N if and only if either $D_1=0$ or $D_2=0$.***

Keywords: * derivation, Γ-nearring, 2-torsion free

1. Introduction

The notion of derivations in nearrings introduced by Bell [1] and Mason. They obtained some basic properties of derivations in nearrings and some commutativity conditions for a Γ- * nearrings with derivations. Cho [2] studied some characterization of Γ- * nearrings and some regularity conditions.

In classical ring theory, Posner [3], Herstein [4], Bergen [5], Bell amd Daif [6] worked on derivations in prime and semiprime rings and gets some commutativity properties of prime rings with derivations. In nearring theory, Bell and Mason [1] and also Cho [7] worked on derivations in prime and semiprime nearrings.

This paper, we slightly extended the results of Cho [7] in prime Γ-* nearrings with certain conditions by using derivations.

2. Main Theorems

Lemma 1: Let D be an arbitrary additive endomorphism of N. Then $D(a\alpha b)= a\alpha D(b)+ D(a)\alpha b^*$ if and only if $D(a\alpha b)= D(a)\alpha b^*+ a\alpha D(b)$ for all a, bϵN and $\alpha\epsilon\Gamma$.

Proof: Assume that $D(a\alpha b)= D(a)\alpha b^*+a\alpha D(b)$ for all a, bϵN and $\alpha\epsilon\Gamma$. For $\alpha\epsilon\Gamma$ and from
$a\alpha(b+b)=a\alpha b+a\alpha b$
and N satisfies left distributive law,
$D(a\alpha(b+b))=a\alpha D(b+b)+D(a)\alpha(b^*+b^*)$
$= a\alpha(D(b)+D(b))+D(a)\alpha b^*+ D(a)\alpha b^*$
$= a\alpha D(b)+a\alpha D(b)+D(a)\alpha b^*+ D(a)\alpha b^*$
and $D(a\alpha b+a\alpha b)= D(a\alpha b)+D(a\alpha b)$
$=D(a)\alpha b^*+ a\alpha D(b)+ D(a)\alpha b^*+ a\alpha D(b)$
By comparing these two equations, we have
$a\alpha D(b)+D(a)\alpha b^*= D(a)\alpha b^*+ a\alpha D(b)$
Hence $D(a\alpha b)= D(a)\alpha b^*+ a\alpha D(b)$ for all a, bϵN and $\alpha\epsilon\Gamma$.
Conversely, assume that $D(a\alpha b)= D(a)\alpha b^*+ a\alpha D(b)$ for all a, bϵN and $\alpha\epsilon\Gamma$
Then form $D(a\alpha(b+b))= D(a\alpha b+a\alpha b)$ and we can induce that, the above calculate of this equality $D(a\alpha b)= a\alpha D(b)+ D(a)\alpha b^*$ for all a, bϵN and $\alpha\epsilon\Gamma$. ◊

Lemma 2: Let D be a * derivation on N. Then N satisfies the following right distributive laws, a, b, cϵN and $\alpha, \beta\epsilon\Gamma$.
$\{a\alpha D(b)+D(a)\alpha b^*\}\beta c = a\alpha D(b)\beta c+D(a)\alpha b^*\beta c$,
$\{D(a)\alpha b^*+ a\alpha D(b)\}\beta c= D(a)\alpha b^*\beta c+a\alpha D(b)\beta c$

Proof: From the calculation for $D((a\alpha b)\beta c)=D(a\alpha(b\beta c))$ for all a, b, cϵN and $\alpha, \beta\epsilon\Gamma$ and Lemma 1, we can find our result.

Lemma 3: Let N be a prime Γ-nearring and U be a nonzero subset of N. If 'a' be an element of N such that $U\Gamma a=\{0\}$, then a=0.

Proof: Since $U\neq\{0\}$, there exists an element uϵU such that $u\neq 0$.Consider that $u\Gamma N\Gamma a\subset U\Gamma a=\{0\}$.Since $u\neq 0$ and N is a prime Γ-nearring, we have that a=0. ◊

Lemma 4: Let N be a prime Γ-nearring and U a nonzero N-subset of N.If D is a nonzero * derivation on N then
(i) If a, bϵN and $a\Gamma U\Gamma b=\{0\}$, then a=0 or b=0
(ii) If aϵN and $D(U)\Gamma a=\{0\}$, then a=0
(iii) If aϵN and $a\Gamma D(U)=\{0\}$, then a=0.

Proof: (i) Let a, bϵN and $a\Gamma U\Gamma b=\{0\}$
Then $a\Gamma U\Gamma N\Gamma b\subset a\Gamma U\Gamma b=\{0\}$.
Since N is a prime Γ-nearring, $a\Gamma U=0$ or b=0
If b=0, then we are done.
So, if $b\neq 0$, then $a\Gamma U=0$
By the lemma 3, then a=0
(ii) Suppose $D(U)\Gamma a=\{0\}$, for aϵN.Then for all uϵU and bϵN, by the lemma, we have
for all a, bϵN and $\alpha, \beta\epsilon\Gamma$, $0=D(b\alpha u)\beta a= (b\alpha D(u)+ D(b)\alpha u^*)\beta a$
$= b\alpha D(u)\beta a+ D(b)\alpha u^*\beta a$
So $D(b)\Gamma U\Gamma a=\{0\}$, for all b$\epsilon$N. (Since $D(U)\Gamma a=\{0\}$)
Since D is a nonzero * derivation on N, we have that a=0 by the statement (i)
(iii) Suppose $a\Gamma D(U)=\{0\}$ for aϵN. Then for all uϵU, bϵN and $\alpha, \beta\epsilon\Gamma$.
$0=a\alpha D(u\beta b)=a\alpha\{ u\beta D(b)+ D(u)\beta b^*\}$
$= a\alpha u\beta D(b)+a\alpha D(u)\beta b^*$
$= a\alpha u\beta D(b)$
Hence $a\Gamma U\Gamma D(b)=\{0\}$ for all bϵN from the statement (i) and D is a nonzero * derivation on N, we have that a=0
We assume that N is a Γ-ring and U is a right N- subset. ◊

Theorem 1: Let N be a prime Γ-nearring and U be a right N-subset of N.If D is a nonzero * derivation on N such that $D^2(u)=0$ then $D^2=0$

Proof: For all u, vϵU and $\alpha\epsilon\Gamma$,
We have $D^2(u\alpha v)=0$.

Then $0= D^2(u\alpha v)= D(D(u\alpha v))=D\{D(u)\alpha v^*+ u\alpha D(v)\}$
$= D^2(u)\alpha(v^*)^*+ D(u)\alpha D(v^*)+ D(u)\alpha D(v^*)+ u\alpha D^2(v)$
Thus $D(U)\Gamma D(U)=\{0\}$ for all $u\epsilon U$.
From lemma 4(iii), we have $D(U)=0$.
Now, for all $b\epsilon N$, $u\epsilon U$ and $\alpha\epsilon\Gamma$
$D^2(u\alpha b)=D(D(a\alpha b))=D(D(u)\alpha b^*+u\alpha D(b))$
$=u\alpha D^2(b)+ D^2(u)\alpha b^*+ D(u)\alpha D(b)+ D(u)\alpha D(b^*)$
Hence $U\Gamma D^2(b)=\{0\}$ for all $b\epsilon N$.
By the lemma 4, we have $D^2(b)=0$ for all $b\epsilon N$. Consequently $D^2=0$. ◊

Lemma 4: Let D be a * derivation of a prime Γ-nearring N and 'a' be an element of N. If $a\alpha D(x)=0$(or $D(x)\alpha a=0$) for all $x\epsilon N$, $\alpha\epsilon\Gamma$, then either a=0 or D=0.

Proof: We assume that $a\alpha D(x)=0$ for all $x\epsilon N$, $\alpha\epsilon\Gamma$.
Change x by $x\beta y$ (for all $\beta\epsilon\Gamma$), we have that
$a\alpha D(x\beta y)=0= a\alpha D(x)\ \beta y^*+ a\alpha x\ \beta D(y)$.
Then by lemma 2,
Since $a\alpha D(x)=0$, $a\alpha x\ \beta D(y)=0$ for all $x, y\epsilon N$, $\alpha, \beta\epsilon\Gamma$.
Since N is a Γ-prime nearring, either a=0 or D=0. ◊

Theorem 2: Let N be a Γ-prime nearring with nonzero * derivation and D_1 and D_2 such that for all $x, y\epsilon N$ and $\alpha\epsilon\Gamma$, $D_1(x)\alpha D_2(y)= - D_2(x)\alpha D_1(y)$. Then N is an abelian Γ-nearring

Proof: Let $x, u, v\epsilon N$, $\alpha\epsilon\Gamma$.
We have $D_1(x)\alpha D_2(y)= - D_2(x)\alpha D_1(y)$
Substitute y by u+v
$0= D_1(x)\alpha D_2(u+v)+ D_2(x)\alpha\ [D_1(u)+D_1(v)]$
$= D_1(x)\alpha\ [D_2(u)+D_2(v)]+ D_2(x)\alpha\ [D_1(u)+D_1(v)]$
$= D_1(x)\alpha D_2(u)+ D_1(x)\alpha D_2(v)- D_1(x)\alpha D_2(u)- D_1(x)\alpha D_2(v)$
$= D_1(x)\alpha\ [D_2(u)+ D_2(v)- D_2(u)- D_2(v)]$
$= D_1(x)\alpha D_2(u+v-u-v)$
Thus $D_1(N)\Gamma D_2(u+v-u-v)=\{0\}$ (1)
By the Lemma 4, we have
$D_2(u+v-u-v)=\{0\}$ (2)
Now, we substitute $x\beta u$ and $x\beta v$ ($\beta\epsilon\Gamma$) instead of u and v respectively in (2) .Then from Lemma 1, we deduce that for all $x, u, v\epsilon N$, $\beta\epsilon\Gamma$.
$0=D_2(x\beta u+ x\beta v- x\beta u- x\beta v)$
$=D_2\ [x\beta(u+v-u-v)]$
$=D_2(x)\beta(u+v-u-v)^*+ x\beta D_2(u+v-u-v)$ (by Lemma 4)
$=D_2(x)\beta(u+v-u-v)^*$
Substitute $(u+v-u-v)^*$ by u+v-u-v.
Again apply by Lemma 4, we see that for all $u, v\epsilon N$, u+v-u-v=0.
$\Rightarrow u+v =v+u$
Consequently, N is an abelian Γ-nearring. ◊

Theorem 3: Let N be a prime Γ-nearring of 2-torsion free and let $D_1$and D_2 be * derivations with the condition $D_1(a)\alpha D_2(b)= D_2(b)\alpha D_1(a)$ for all $a, b\epsilon N$, $\alpha\epsilon\Gamma$ on N .Then D_1D_2 is a * derivation on N if and only if either $D_1=0$, $D_2=0$.

Proof: We assume that D_1D_2 is a * derivation. Then we get for $\alpha\epsilon\Gamma$,
$D_1D_2(a\alpha b)=a\alpha D_1D_2(b)+D_1D_2(a)\alpha b^*$ (3)
Also, since D_1 and D_2 are * derivations, we obtain
$D_1D_2(a\alpha b)= D_1(D_2(a\alpha b))= D_1(a\alpha D_2(b)+D_2(a)\alpha b^*)$
$= D_1(a\alpha D_2(b))+ D_1(D_2(a)\alpha b^*)$
$= a\alpha D_1D_2(b)+D_1(a)\alpha D_2(b^*)+ D_2(a)\alpha D_1(b^*)^*+ D_1D_2(a)\alpha b^*$
$= a\alpha D_1D_2(b)+D_1(a)\alpha D_2(b^*)+ D_2(a)\alpha D_1(b)+ D_1D_2(a)\alpha b^*$ (4)
From (3) and (4) for $D_1D_2(a\alpha b)$ for all $a, b\epsilon N$, $\alpha\epsilon\Gamma$,
$D_1(a)\alpha D_2(b^*)+D_2(a)\alpha D_1(b)=0$ (5)
Hence from Theorem 2, we know that N is an abelian Γ-nearring.
Substitute a by $a\alpha D_2(c)$ in (5) and using Lemma 1 and * derivations in right distributive laws, we get that
$0= D_1(a\alpha D_2(c))\alpha D_2(b^*)+ D_2(a\alpha\ D_2(c))\alpha D_1(b)$
$=\{D_1(a)\alpha D_2(c^*)+a\alpha D_1D_2(c)\}\alpha D_2(b^*)+ \{a\alpha D^2{}_2(c^*)+ D_2(a)\alpha D_2(c)\}\alpha D_1(b)\}$
$=D_1(a)\alpha D_2(c^*)\ \alpha D_2(b^*)+a\alpha D_1D_2(c)\alpha D_2(b^*)+ a\alpha D^2{}_2(c^*)\ \alpha D_1(b)+ D_2(a)\alpha D_2(c)\alpha D_1(b$ $=D_1(a)\alpha D_2(c^*)\ \alpha D_2(b^*)+ a\alpha\{D_1D_2(c)\alpha D_2(b^*)+ D^2{}_2(c^*)\ \alpha D_1(b)\}+D_2(a)\alpha D_2(c)\alpha D_1(b)$
Again change a by $D_2(c)$ in (5), we see that
$D_1(D_2(c)\alpha D_2(b^*))+ D_2(D_2(c))\alpha D_1(b)=0$
This implies that $a\alpha\{\ D_1D_2(c)\alpha D_2(b^*)+ D^2{}_2(c))\alpha D_1(b)\}=0$
Hence, from the above long equality, we have the following equality,
$D_1(a)\alpha D_2(c^*)\ \alpha D_2(b^*)+ D_2(a)\alpha D_2(c)\ \alpha D_1(b)=0$, for all $a, b\epsilon N$, $\alpha\epsilon\Gamma$. (6)
Replace a and b by c in (5), we get
$D_2(c)\ \alpha D_1(b)= - D_1(c)\ \alpha D_2(b^*)$, $D_1(a)\alpha D_2(c^*)= -D_2(a)\ \alpha D_1(c)$
So that (6) becomes,
$0= \{-D_2(a)\ \alpha D_1(c^*)\}\alpha\ D_2(b^*)+ D_2(a)\alpha\{\ -D_1(c)\ \alpha D_2(b^*)\}$
$=D_2(a)\ \alpha(-D_1(c^*))\alpha\ D_2(b^*)+ D_2(a)\alpha(\ -D_1(c))\alpha D_2(b^*)$
$=D_2(a)\beta\{-D_1(c^*))\alpha\ \ D_2(b^*)-D_1(c)\alpha D_2(b^*)\}$, for all $a, b\epsilon N$, $\alpha\epsilon\Gamma$.
If $D_2\neq 0$ then by Lemma 4,
We have the equality, $(-D_1(c^*))\alpha\ D_2(b^*)\ -D_1(c)\alpha D_2(b^*)=0$.
That is $D_1(c^*)\alpha D_2(b^*)=(-D_1(c))\alpha D_2(b^*)$, for all $b, c\epsilon N$, $\alpha\epsilon\Gamma$. (7)
Thus, using this condition of our theorem, we obtain
$(-D_1(c))\alpha D_2(b^*)=D_1(-c)\alpha D_2(b^*)=-D_2(b^*))\alpha D_1(-c)=\ \ D_2(b^*)\alpha(-D_1(c))= -D_2(b^*)\alpha D_1(c)$
$=-D_1(c))\alpha D_2(b^*)$ (8)
From (7) and (8) we have that, for all $b, c\epsilon N$, $\alpha\epsilon\Gamma$,
$2D_1(c)\alpha D_2(b^*)=0$.
Since N is 2-torsion free, $D_1(c)\alpha D_2(b^*)=0$
Also, since D_2 is not zero, by Lemma 4, we see that $D_1(c)=0$ for all $c\epsilon N$. Therefore $D_1=0$
Consequently, either $D_1=0$ or $D_2=0$.
The converse verification is obvious ◊.

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