

# Euler - Darboux Equation Involving Exp x of Convolution Type-I

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**Abstract:** In this paper, the authors have established the solution of Euler-Darboux equation involving exponential functions of convolution type I by using fractional integration. Result is useful and plays important role in physical & mathematical sciences.

**Keywords:** Euler-Darboux equation, fractional integral operator, Gauss hypergeometric function, Hölder Continuity, singularity of function

## 1. Introduction

Recently Authors in paper [1] have defined generalized Lowndes’s operator and Hankel operator involving with exponential function. In another paper Authors in paper [2] discussed the solution of certain special quadruple integral equations associated with multi-dimensional inverse Mellin transform of given function by the application of extended form of Erdélyi—Kober operators.

In this paper authors have discussed Euler-Darboux equation associated with exponential function of convolution type-I by following the method due to [6, 7, 8, 9]. In the section 2 definition of the fractional integrals and derivatives are given with several properties which have been used frequently in this paper.

The Euler Darboux equation

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\beta}{e^x - e^y} \frac{\partial u}{\partial x} + \frac{\alpha}{e^x - e^y} \frac{\partial u}{\partial y} = 0 \quad (\alpha > 0, \beta > 0, \alpha + \beta < 1) \tag{1}$$

implies degenerate exponential equations expressed by the characteristic coordinates. So in section (3), we set a problem for equation (1) with boundary conditions on two characteristics that contain our fractional integral or derivative of order less than some members depending upon  $\alpha, \beta$ , where the problem may be regarded as a generalization of the Gour-sat problem, i.e. initial value problem for equation (1). Then there will be given an expression of a solution for our problem. To see this the problem will be reduced to a dominant singular integral equation with Cauchy Kernel, where several calculations will be carried out in section (4).

## 2. Generalized Fractional Integral and Derivative

Let  $\alpha > 0$ , and  $\beta, \eta, \delta$  and  $t$  be real numbers. We shall define a fractional integral in paper [5] of real and continuous function  $f(x)$  on  $(a, \infty)$  which may have an infinity at  $x = a$  of order less than 1 or  $-\beta + \eta + 1$  if  $\beta < \eta$  or  $\beta > \eta$ , respectively, by

$$I_{ax}^{\alpha, \beta, \eta} f \equiv \frac{(\exp x - \exp a)^{-\alpha - \beta}}{\Gamma(\alpha)} \int_a^x (\exp x - \exp t)^{\alpha - 1} f(t) dt$$

$$F \left( \alpha + \beta, -\eta; \alpha; \frac{\exp x - \exp t}{\exp x - \exp a} \right) \tag{2}$$

Where  $\Gamma$  is the gamma function,  $F$  means the Gauss hypergeometric function and

$$F(a; b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

in  $|z| < 1$  and its analytic continuation into  $|\arg z| < \pi$  and

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}$$

The expression (2) is a generalization of the fractional integrals of both Riemann - Liouville, Erdelyi-Kober, Hardy and Littlewood i.e.

$$I_{ax}^{\alpha, -\alpha, \eta} f = \frac{1}{\Gamma(\alpha)} \int_a^x (\exp x - \exp t)^{\alpha - 1} f(t) dt \equiv R_{ax}^{\alpha} f \tag{3}$$

And

$$I_{ax}^{\alpha, 0, \eta} f = \frac{(\exp x - \exp a)^{-\alpha - \eta}}{\Gamma(\alpha)} \int_a^x (\exp x - \exp t)^{\alpha - 1} \cdot (\exp t - \exp a)^{\eta} f(t) dt \equiv E_{ax}^{\alpha, \eta} f \tag{4}$$

respectively. In paper [5], for  $\alpha < 0$  a generalized fractional derivative is defined by

$$I_{ax}^{\alpha, \beta, \eta} f \equiv \frac{d^n}{dx^n} I_{ax}^{\alpha + n, \beta - n, \eta - n} f, \tag{5}$$

If the right hand side has a definite meaning, where it is assumed that  $0 < \alpha + \eta \leq 1$  and  $n$  is positive integer. The relation (5) also valid for  $\alpha > 0$ .

A fractional integral whose lower limit is a variable  $x$  smaller than the constant upper limit  $b$  is defined by

$$J_{xb}^{\alpha, \beta, \eta} f \equiv I_{0, b-x}^{\alpha, \beta, \eta} f_1 \equiv \frac{(\exp b - \exp x)^{-\alpha - \beta}}{\Gamma(\alpha)} \int_x^b (\exp t - \exp x)^{\alpha - 1} F \left[ \begin{matrix} x \\ x \end{matrix} \right] f(t) dt \tag{6}$$

where

$$F \begin{bmatrix} x \\ x \end{bmatrix} = F \left( \alpha + \beta, -\eta; \alpha; \frac{\exp t - \exp x}{\exp b - \exp x} \right)$$

Here  $\alpha > 0$  and  $f_1(t) = f(b-t)$ . If  $0 < \alpha + n \leq 1$ , we shall define a fractional derivative by

$$J_{xb}^{\alpha, \beta, \eta} f \equiv (-1)^n \frac{d^n}{dx^n} J_{xb}^{\alpha+n, \beta-n, \eta-n} f, \quad (7)$$

where  $n$  is a positive integer. Let us write

$$J_{xb}^{\alpha, -\alpha, \eta} \equiv L_{xb}^{\alpha} f.$$

As found in paper [5], the following product rules hold:

$$I_{ax}^{\alpha, \beta, \eta} I_{ax}^{\gamma, \delta, \alpha+\eta} = I_{ax}^{\alpha+\gamma, \beta+\delta, \eta} \quad (\alpha, \gamma > 0) \quad (8)$$

$$I_{ax}^{\alpha, \beta, \eta} I_{ax}^{\gamma, \delta, \eta-\beta-\gamma-\delta} = I_{ax}^{\alpha+\gamma, \beta+\delta, \eta-\gamma-\delta} \quad (\alpha, \gamma > 0) \quad (9)$$

These are still valid for negative non-integer  $\alpha$ .

It is easily seen that  $I_{ax}^{0,0,\eta}$  is the identity operator for any  $h$ ,

then the inverse operator of  $I_{ax}^{\alpha, \beta, \eta}$  is given by :

$$\left( I_{ax}^{\alpha, \beta, \eta} \right)^{-1} = I_{ax}^{-\alpha, -\beta, \alpha+\eta} \quad (10)$$

which follow from formulas (8) or (9)

The formulas (8) to (10) hold true also for the operator  $J$  defined by equation (6).

### 3. The Euler-Darboux Equation

Consider the equation (1) in the domain  $\Omega = (0 < x < y < 1)$ . It is well known in paper [10] and Book [11] that the solution of equation (1) under the conditions.

$$u|_{y=x} = \tau(x), (\exp y - \exp x)^{\alpha+\beta} \left( \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \right) |_{y=x} = v(x) \quad (11)$$

is expressed as :

$$u(x, y) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \tau[\exp x + (\exp y - \exp x) \exp t] (\exp t)^{\beta-1} (1 - \exp t)^{\alpha-1} dt + \frac{\Gamma(1 - \alpha - \beta)}{2\Gamma(1 - \alpha)\Gamma(1 - \beta)} (\exp y - \exp x)^{1-\alpha-\beta} \int_0^1 v[\exp x + (\exp y - \exp x) \exp t] (\exp t)^{-\alpha} (1 - \exp t)^{-\beta} dt \quad (12)$$

Then the values of  $u$  on two characteristics  $x=0$  and  $y=1$  can be written respectively in terms of the fractional integrals (2) and (6) as follows:

$$u^{(1)}(y) \equiv u(0, y) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} I_{0y}^{\alpha, 0, \beta-1} \tau + \frac{\Gamma(1 - \alpha - \beta)}{2\Gamma(1 - \alpha)} I_{0y}^{1-\beta, \alpha+\beta-1, \beta-1} v, 0 < y < 1, \quad (13)$$

$$u^{(2)}(x) \equiv u(x, 1) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} J_{x1}^{\beta, 0, \alpha-1} \tau + \frac{\Gamma(1 - \alpha - \beta)}{2\Gamma(1 - \beta)} J_{x1}^{1-\alpha, \alpha+\beta-1, \alpha-1} v, 0 < x < 1 \quad (14)$$

Now we shall set a problem which is the main subject for the rest of this work.

Let  $H^k(T)$  be a class of functions which are defined on a real interval  $T$  and Hölder continuous in  $T$  with the Hölder index  $k$ . Let us denote the open interval  $(0, 1)$  by  $U$  and its closure by  $\bar{U}$ .

**Problem A :** To seek for a solution  $u(x, y)$  of equation (1) in  $\Omega$  such that

(i)  $\tau(x) \in H^{k_1}(\bar{U})$  and  $v(x) \in H^{k_2}(U)$  for some  $k_1$ , and  $k_2$  ( $0 < k_1, k_2 < 1$ )

where  $v(x)$  may have infinites of order not greater than  $1 - \alpha - \beta$  at the end point of  $U$ .

(ii)  $u^{(1)}(y)$  and  $u^{(2)}(x)$  in (13) and (14) satisfy respectively the boundary conditions

$$I_{0y}^{a, b, -a+\beta-1} u^{(1)} = \varphi_1(y), y \in U \quad \text{and} \quad (15)$$

$$J_{x1}^{c, d, -c+\alpha-1} u^{(2)} = \varphi_2(x), x \in U, \quad (16)$$

where  $a, b, c$  and  $d$  are constants such that  $\max(-\alpha, \beta-1) < a < \min(\beta, 1-\alpha)$ ,  $b > -\alpha-\beta$ ,  $\max(-\beta, a-1) < c < \min(\alpha, 1-\beta)$  and  $d > -\alpha-\beta$ , and  $\varphi_1 \in H^{k_3}(\bar{U})$  [ $\max(a-\beta+1, a+\alpha, -b) < k_3 < 1$ ,  $\varphi_1(0)=0$ ] and  $\varphi_2 \in H^{k_4}(\bar{U})$  [ $\max(c-\alpha+1, c+\beta, -d-\alpha-\beta+1) < k_4 < 1$ ,  $\varphi_2(1) = 0$ ] are given functions.

Note that if  $a = b = c = d = 0$ , problem A is reduced to the Goursat problem. If we note (13), (14) and (8), and replace  $y$  by  $x$ , the conditions (15) and (16) may be read as :

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} I_{0x}^{a+\alpha, b, -a+\beta-1} \tau + \frac{\Gamma(1 - \alpha - \beta)}{2\Gamma(1 - \alpha)} I_{0x}^{a-\beta+1, b+\alpha+\beta-1, -a+\beta-1} v = \varphi_1(x) \quad (17)$$

and

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} I_{x1}^{c+\beta, d, -c+\alpha-1} \tau + \frac{\Gamma(1 - \alpha - \beta)}{2\Gamma(1 - \beta)} J_{x1}^{c-\alpha+1, d+\alpha+\beta-1, -c+\alpha-1} v = \varphi_2(x) \quad \text{on } U \quad (18)$$

thus problem (A) is reduced to

**Problem (B):** To seek for solutions  $\tau(x)$  and  $v(x)$  of (17) and (18) satisfying the condition (i) in problem A, where  $\varphi_1(x)$ ,  $\varphi_2(x)$ ,  $a, b, c$  and  $d$  are as given in problem A.

Operating  $\left( I_{0x}^{a+\alpha, b, -a+\beta-1} \right)^{-1} = I_{0x}^{-a-\alpha, -b, \alpha+\beta-1}$  on both sides of equation (17), we have

$$\tau(x) + \frac{\Gamma(\beta)\Gamma(1 - \alpha - \beta)}{2\Gamma(\alpha + \beta)\Gamma(1 - \alpha)} R_{0x}^{1-\alpha-\beta} v = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} I_{0x}^{-a-\alpha, -b, \alpha+\beta-1} \varphi_1 \quad (19)$$

Similarly, by using  $\left( J_{0x}^{c+\beta, d, -c+\alpha-1} \right)^{-1} = J_{0x}^{-c-\beta, -d, \alpha+\beta-1}$  we have

$$\tau(x) + \frac{\Gamma(\alpha)\Gamma(1-\alpha-\beta)}{2\Gamma(\alpha+\beta)\Gamma(1-\alpha)} L_{x1}^{1-\alpha-\beta} v = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} J_{x1}^{-c-\beta,-d,\alpha+\beta-1} \varphi_2 \quad (20)$$

The subtraction of equation (20) from equation (19) and the operation of  $R_{0x}^{\alpha+\beta-1}$  on both sides given the relation,

$$v(x) - \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{2\Gamma(\beta)\Gamma(1-\beta)} R_{0x}^{\alpha+\beta-1} L_{x1}^{1-\alpha-\beta} v = \Phi_0(x) \quad (21)$$

where,

$$\Phi_0(x) = \frac{2\Gamma(1-\alpha)}{\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1} I_{0x}^{-a-\alpha,-b,\alpha+\beta-1} \varphi_1 - \frac{2\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\beta)\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1} J_{x1}^{-c-\beta,-d,\alpha+\beta-1} \varphi_2 \quad (22)$$

It is well known in paper [10] and Book [11], the relation

$$R_{0x}^{-\alpha} L_{x1}^{\alpha} \varphi = \cos \pi \alpha \varphi(x) + \frac{\sin \pi \alpha}{\pi} \int_0^1 \left( \frac{\exp u}{\exp x} \right)^{\alpha} \frac{1}{\exp u - \exp x} \varphi(u) du$$

hold valid for  $0 < \alpha < 1$ , where the integral is taken in the sense of the Cauchy principal value. Then equation (21) may be written in the form

$$v(x) - \frac{\tan \pi \beta}{\pi} \int_0^1 \left( \frac{\exp u}{\exp x} \right)^{1-\alpha-\beta} \frac{1}{\exp u - \exp x} v(u) du = \frac{\sin \pi \alpha}{\cos \pi \alpha \sin \pi(\alpha + \beta)} \Phi_0(x) \quad (23)$$

Here we have used the relation  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ , due to Book [4].

If we use a new unknown function  $\mu(x) = v(x) x^{1-\alpha-\beta}$ , then we have from equation (23) the dominant singular integral equation for  $\mu(x)$ .

$$\mu(x) - \frac{\tan \pi \alpha}{\pi} \int_0^1 \frac{\mu(u)}{\exp u - \exp x} du = \Phi(x) \text{ on } U, \quad (24)$$

where

$$\Phi(x) = \frac{2\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\cos \pi \beta)} (\exp x)^{1-\alpha-\beta} R_{0x}^{\alpha+\beta-1} I_{0x}^{-a-\alpha,-b,\alpha+\beta-1} \varphi_1 - \frac{2\Gamma(\alpha+\beta)}{\Gamma(\beta)(\cos \pi \beta)} (\exp x)^{-1-\alpha-\beta} R_{0x}^{\alpha+\beta-1} J_{x1}^{-c-\beta,-d,\alpha+\beta-1} \varphi_2 \quad (25)$$

Now let us solve equation (24) by applying the theory in Gakhov [3]. To this end we shall summarize results of the theory in an applicable form to our equation (24). Let  $L = t_0 t_1$  be on open non intersecting smooth curve in the complex plane. Functions  $a(t)$ ,  $b(t)$  and  $f(t)$  are assumed to be Hölder

continuous on the closure of  $L$  with  $a^2(t) - b^2(t) = 1$ . Then a solution of the dominate singular integral equation.

$$a(t)\varphi(t) + \frac{b(t)}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - \exp t} d\tau = f(t) \text{ on } L$$

is given by

$$\varphi(t) = a(t)f(t) - \frac{b(t)Z(t)}{\pi i} \int_L \frac{f(\tau)}{Z(\tau)} \frac{d\tau}{(\tau - \exp t)}$$

where

$$Z(t) = \exp \left\{ \frac{1}{2\pi i} \int_L \frac{1}{\tau - \exp t} \text{Log } G(\tau) d\tau \right\}$$

$$\text{and } G(t) = \frac{a(t) - b(t)}{a(t) + b(t)}$$

The above results are introduced under such assumptions that the index  $k$  of the equation is equal to zero and the solution  $\varphi(t)$  is sought to be Hölder continuous on  $L$ , bounded at  $t = t_0$  and unbounded but having integrable order at  $t = t_1$ , where, if we set  $G(t_0) = \rho \exp(i\theta)$  and  $G(t_1) = \rho' \exp(i(\theta + \Delta))$ ,  $\theta$  and  $k$  are chosen such as  $-2\pi < \theta \leq 0$  and  $k = [(\theta + \Delta)/2\pi] + 1$ . Here  $\Delta$  is the change of  $\arg G(t)$  on  $L$  and  $[x]$  denotes the greatest integer not exceeding  $x$ .

Returning to our equation (24), we have

$$G(x) = \frac{1 + i \tan \pi \beta}{1 - i \tan \pi \beta} = e^{2\pi i(\beta-1)}$$

Then  $\theta = 2\pi(\beta-1)$ ,  $\Delta = 0$  and  $k = 0$  are obtained. Therefore the solution of equation (24) is written in the form

$$\mu(x) = \frac{1}{1 + \tan^2 \pi \beta} \left[ \Phi(x) + \frac{\tan \pi \beta}{\pi} \left( \frac{\exp x}{1 - \exp x} \right)^{1-\beta} \int_0^1 \left( \frac{\tau - \exp u}{\exp u} \right)^{1-\beta} \frac{1}{\exp u - \exp x} \Phi(u) du \right]$$

Which is Hölder continuous in  $U$ , bounded at  $x = 0$  and unbounded but having integrable singularity at  $x = 1$ . Here we have taken into consideration that  $\Phi(x)$  is Hölder continuous on  $\bar{U}$ , whose continuity will be proved in the next section.

From above we may easily find  $v(x)$  satisfying the condition (i). Since it will be seen in the next section that  $v(x)$  is of order greater than  $\alpha + \beta - 1$  at  $x = 1$ . Then in view of equations (19) or (20)  $\tau(x)$  satisfying (i) is determined.

#### 4. Regularity of $\Phi(x)$ and singularity of $v(x)$

We need the following Lemmas.

**Lemma 1:** Let  $a < b$  and  $0 < k < 1$ . If  $\varphi, \psi \in H^k[a, b]$  with  $\varphi(a) = \psi(b) = 0$ ,  $0 > \alpha > -k$ ,  $\beta < k$  and  $\eta > \beta - 1$ , then  $I_{ax}^{\alpha,\beta,\eta} \varphi, J_{xb}^{\alpha,\beta,\eta} \psi \in H^{k+\alpha}[a, b]$ .

**Proof:**

As in easily calculated, we have

$$I_{ax}^{\alpha,\beta,\eta} \varphi = \frac{d}{dx} I_{ax}^{\alpha+1,\beta-1,\eta-1} \varphi = \frac{\Gamma(-\beta + \eta + 1)}{\Gamma(-\beta + 1)\Gamma(\alpha + \eta + 1)} \left[ \left( 1 - \frac{\exp h}{\exp x - \exp a + \exp h} v \right)^q dv \right] = 0 (\exp h)^{k+\alpha} \quad (29)$$

$$(\exp x - \exp a)^{-\beta} \varphi(x) + \frac{(\exp x - \exp a)^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (\exp x - \exp t)^{\alpha-1} F\left(\alpha + \beta, \eta; \alpha; \frac{\exp x - \exp t}{\exp x - \exp a}\right) \{\varphi(t) - \varphi(x)\} dt \quad (26)$$

It is well known in Book [4] that the hypergeometric function  $F(a,b;c;z)$  has orders such that  $O(1)$ , ( $z \rightarrow 0$ );  $O[(1-z)^{\min(0, c-a-b)}]$ , ( $z \rightarrow 1$ ), then the kernel of integral in equation (26) has the properties :

$$(\exp x - \exp t)^{\alpha-1} F\left(\alpha + \beta, \eta; \alpha; \frac{\exp x - \exp t}{\exp x - \exp a}\right) = O[(\exp x - \exp t)^{\alpha-1}](t \rightarrow x)$$

and  $O[(\exp t - \exp a)^{\min(0, -\beta+\eta)}]$ , ( $t \rightarrow a$ )

Then in order to prove the Hölder continuity of  $I_{ax}^{\alpha,\beta,\eta} \varphi$  it is enough to show that

$$A(x) = \int_a^x (\exp t - \exp a)^q (\exp x - \exp t)^{\alpha-1} \{\varphi(t) - \varphi(x)\} dt \in H^{k+\alpha} [a, b] \quad (27)$$

where  $q = \min(0, -\beta + \eta)$ . Let  $a \leq x < x + h \leq b$  and consider the difference

$$A(x+h) - A(x) = \int_a^{x+h} (\exp t - \exp a)^q (\exp x + \exp h - \exp t)^{\alpha-1} [\varphi(t) - \varphi(x+h)] dt - \int_a^x (\exp t - \exp a)^q (\exp x - \exp t)^{\alpha-1} [\varphi(t) - \varphi(x)] dt$$

$$= \int_a^x (\exp t - \exp a)^q [(\exp x + \exp h - \exp t)^{\alpha-1} - (\exp x - \exp t)^{\alpha-1}] [\varphi(t) - \varphi(x)] dt + \int_a^{x+h} (\exp t - \exp a)^q (\exp x + \exp h - \exp t)^{\alpha-1} [\varphi(t) - \varphi(x+h)] dt = I_1 + I_2 + I_3 \quad (28)$$

Changing the variables of integration suitably, we have as  $h \rightarrow 0$

$$I_1 = 0 \left[ \int_a^x (\exp t - \exp a)^q (\exp x - \exp t)^k \left\{ (\exp x + \exp h - \exp t)^{\alpha-1} - (\exp x - \exp t)^{\alpha-1} \right\} dt \right]$$

$$= 0 \left[ (\exp h)^{k+\alpha} (\exp x - \exp a + \exp h)^q \int_1^{1+\frac{x-a}{h}} (v-1)^k \{v^{\alpha-1} - (v-1)^{\alpha-1}\} dv \right]$$

$$I_2 = 0 \left[ (\exp x - \exp a)^{q+1} (\exp h)^k (\exp x - \exp a + \exp h)^{\alpha-1} \int_0^1 v^q \left( 1 - \frac{\exp x - \exp a}{\exp x - \exp a + \exp h} v \right)^{\alpha-1} dv \right]$$

$$= 0 \left[ (\exp h)^k F\left(q+1, 1-\alpha; q+2; \frac{\exp x - \exp a}{\exp x - \exp a + \exp h}\right) \right] = 0 (\exp h)^{k+\alpha} \quad (30)$$

$$I_3 = 0 \left[ \int_x^{x+h} (\exp t - \exp a)^q (\exp x + \exp h - \exp t)^{k+\alpha-1} dt \right]$$

$$= 0 \left[ (\exp h)^{h+\alpha} (\exp x - \exp a + \exp h)^q \int_0^1 v^{k+\alpha-1} \left( 1 - \frac{\exp h}{\exp x - \exp a + \exp h} v \right)^q dv \right]$$

$$= 0 \left[ (\exp h)^{k+\alpha} F\left(k+\alpha, -q; k+\alpha+1; \frac{\exp h}{\exp x - \exp a + \exp h}\right) \right] = 0 \left[ (\exp h)^{k+\alpha} \right] \quad (31)$$

Since  $k + \alpha > 0$ ,  $q > -1$  and

$$\int_0^1 (\exp t)^{a-1} (1 - \exp t)^{c-\alpha-1} \{1 - (\exp z)(\exp t)\}^{-b} dt = \Gamma(a) \frac{\Gamma(c-a)}{\Gamma(c)} \Gamma(a, b; c; \exp z)$$

[ $\text{Re } c > \text{Re } a > 0$ ,  $|\arg(1-z)| < \pi$ ,  $z \neq 1$ ] in paper [4].

When equations (28) to (31) are used, equation (27) can be reading proved, proof of  $J_{xb}^{q,\beta,\eta} \Psi \in H^{k+\alpha} [a, b]$  are parallel.

**Lemma 2:** Under the condition of Lemma 1 except  $\varphi(a) = \Psi(b) = 0$  and  $\beta < k$ ,  $(x - a)^\beta I_{ax}^{\alpha,\beta,\eta} \varphi$  and  $(b-x)^\beta J_{xb}^{q,\beta,\eta} \Psi$  belong to  $H^{k+\alpha} [a, b]$ .

**Proof :** The first results immediate by noting the relation (26) and the proof of Lemma 1. The second is similar. Now let us show that  $\Phi(x)$  in (25) is Hölder continuous on  $\bar{U}$ . From Lemma 1 we can easily find that  $g(x) \equiv I_{ax}^{-a-\alpha, -b, \alpha+\beta-1} \varphi \in H^{k_3-a-\alpha}(\bar{U})$ , since  $k_3 > a + \alpha$ ,  $k_3 + b > 0$ ,  $b + \alpha + \beta > 0$  and  $\varphi_1(0) = 0$ . Then Lemma 2 implies  $x^{1-\alpha-\beta} R_{ox}^{\alpha+\beta-1} g \in H^{k_3-a+\beta-1}(\bar{U})$ . Similarly Lemma 1 guarantees that  $h(x) \equiv J_{x1}^{-c-\beta, -d, \alpha+\beta-1} \varphi_2 \in H^{k_4-c-\beta}(\bar{U})$  by virtue of the as-

sumptions  $k_4 > c + \beta$ ,  $k_4 + d > 0$ ,  $d + \alpha + \beta > 0$  and  $\varphi_2(1) = 0$ . Thus from Lemma 2  $x^{1-\alpha-\beta} R_{0x}^{\alpha+\beta-1} h \in H^{k_4-c+d-1}(\bar{U})$ . Therefore it has been proved that  $\Phi(x) \in H^{\min(K_3-a+\beta-1, K_4-c+\alpha-1)}(\bar{U})$ .

[11] M.M. Smirnov, Degenerate Hyperbolic equations, Izdat. Vyseisaja Skola, Minslc. 1977.

To see the order of singularity of  $v(x)$  at  $x = 1$  we operate  $(J_{x1}^{c-\alpha+1, d+\alpha+\beta-1, -c+\alpha-1})^{-1} = (J_{x1}^{-c-\alpha-1, -d-\alpha-\beta+1, 0})$  on both sides of equation (18). Then we have

$$v(x) = \frac{2\Gamma(\alpha + \beta)\Gamma(1 - \beta)}{\Gamma(\alpha)\Gamma(1 - \alpha - \beta)} L_{x1}^{\alpha+\beta-1} \tau + \frac{2\Gamma(1 - \beta)}{\Gamma(1 - \alpha - \beta)} J_{x1}^{-c+\alpha-1, -d-\alpha-\beta+1, 0} \varphi_2 \quad (32)$$

Thus Lemma 1 implies that  $v(x) = O[(1 - x)^{\min(k_1 + \alpha + \beta - 1, k_4 - c + \alpha - 1)}]$  ( $x \rightarrow 1$ ) where  $k_1$  is the Hölder index of  $\tau$ . Here the condition  $\tau(1) = 0$  has been used, which following by virtue of equation (20) and Lemma 1. Hence we have established that  $v$  is of order greater than  $\alpha + \beta - 1$  at  $x = 1$ .

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