\(\xi\)-Normal and \(\xi\)-Regular Spaces in Topological Spaces

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Abstract: The aim of this paper is to introduce and study two new classes of spaces, namely \(\xi\)-normal and \(\xi\)-regular spaces in topological spaces. The relationships among normal, \(p\)-normal, \(\alpha\)-normal, \(\beta\)-normal and \(\xi\)-normal spaces are investigated. Moreover, we introduced some functions such as \(g\xi\)-closed, \(\xi\)-\(g\xi\)-closed, pre \(\xi\)-open. We obtained some characteristics of \(\xi\)-normal and \(\xi\)-regular spaces, properties of the forms of \(g\xi\)-closed functions and preservation theorems for \(\xi\)-normal and \(\xi\)-regular spaces.

Keywords: \(\xi\)-closed sets, \(\xi\)-normal, \(\xi\)-regular spaces, \(g\xi\)-closed and \(\xi\)-\(g\xi\)-closed functions

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1. Introduction

Levine [3] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. \(\alpha\)-open sets were introduced by Njastad [7]. Devi et al. [2] introduced the concept of \(\xi\)-closed sets. Nour [8] introduced the notion of \(p\)-normal spaces and obtained their characterizations and preservation theorems. Paul and Bhattacharyya [9] obtained some properties of \(p\)-normal spaces. Benchalli et al. [1] introduced the notion of \(\alpha\)-normal spaces and obtained their characterizations and preservation theorems. Mahmoud et al. [4] introduced the notion of \(\beta\)-normal spaces and obtained their characterizations and preservation theorems. Recently, Sharma et al. [10] introduced a new class of regular spaces called \(\xi\)-regular spaces by using \(\xi\)-open sets introduced by Devi et al. [2] and obtained several properties such as characterizations and preservation theorems for \(\xi\)-regular spaces.

2. Preliminaries

Throughout this paper, spaces \((X, \tau)\), \((Y, \sigma)\), and \((Z, \gamma)\) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let \(A\) be a subset of a space \(X\). The closure of \(A\) and interior of \(A\) are denoted by \(\text{Cl}(A)\) and \(\text{Int}(A)\) respectively. \(A\) is said to be \(\alpha\)-open [1] if \(A \subset \text{Int(Cl}(\text{Int}(A)))\). The complement of a \(\alpha\)-open set is said to be \(\alpha\)-closed [1]. The intersection of all \(\alpha\)-closed sets containing \(A\) is called \(\alpha\)-closure [2] of \(A\), and is denoted by \(\text{Cl}(A)\).

2.1 Definition. A subset \(A\) of a space \((X, \tau)\) is said to be

1. \textbf{generalized closed} (briefly \textit{g-closed}) [3] if \(\text{Cl}(A) \subset U\) whenever \(A \subset U\) and \(U \in \tau\).

2. \textbf{generalized} \(\alpha\)-\textit{closed} (briefly \textit{ag-closed}) [6]) if \(\alpha\text{Cl}(A) \subset U\) whenever \(A \subset U\) and \(U \in \tau\).

3. \textbf{generalized} \(\alpha\)-\textit{closed} (briefly \textit{ag-closed}) [5] if \(\alpha\text{Cl}(A) \subset U\) whenever \(A \subset U\) and \(U \in \tau\).

4. \textbf{\(\xi\)-closed} [2] if \(\alpha\text{Cl}(A) \subset U\) whenever \(A \subset U\) and \(U \in \text{g-open in } X\).

5. \textbf{\(g\)-open} (resp. \textit{ag-open}, \textit{ga-open}, \textit{\(\xi\)-open}) if the complement of \(A\) is \(g\)-closed (resp. \(ag\)-closed, \(ga\)-closed, \(\xi\)-closed).

The intersection of all \(\xi\)-closed sets containing \(A\) is called \(\xi\)-\textit{closure} of \(A\), and is denoted by \(\xi\text{Cl}(A)\). The \(\xi\)-\textit{interior} of \(A\), denoted by \(\xi\text{Int}(A)\), is defined as union of all \(\xi\)-open sets contained in \(A\). The family of all \(\xi\)-closed (resp. \(\xi\)-open) sets of a space \(X\) is denoted by \(\xi\text{Cl}(X)\) (resp. \(\xi\text{O}(X)\)).

2.2 Lemma. Let \(A\) be a subset of a space \(X\) and \(x \in X\). The following properties hold for \(\xi\text{Cl}(A)\):

(i) \(x \in \xi\text{Cl}(A)\) if and only if \(A \cap U \neq \emptyset\) for every \(U \in \xi\text{O}(X)\) containing \(x\).

(ii) \(A\) is \(\xi\)-closed if and only if \(A = \xi\text{Cl}(A)\).

(iii) \(\xi\text{Cl}(A) \subseteq \xi\text{Cl}(B)\) if \(A \subset B\).

(iv) \(\xi\text{Cl}(\xi\text{Cl}(A)) = \xi\text{Cl}(A)\).

(v) \(\xi\text{Cl}(A)\) is \(\xi\)-closed.

2.3 Definition. A subset \(A\) of a space \(X\) is said to be \textbf{generalized \(\xi\)-closed} (briefly \textit{g\(\xi\)-closed}) if \(\xi\text{Cl}(A) \subset U\) whenever \(A \subset U\) and \(U \in \tau\).

2.4 Remark. We have the following implications for the properties of subsets:

\[
\text{closed} \Rightarrow \text{g-closed} \\
\text{ag-closed} \Rightarrow \text{ag-closed} \\
\xi\text{-closed} \Rightarrow \text{g}\xi\text{-closed}
\]

Where none of the implications is reversible as can be seen from the following examples:

2.5 Example. Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, X\}\). Then \(A= \{b\}\) is \(g\)-closed but not closed.
2.6 Example. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $A = \{a, b\}$ is $g$-closed as well as $g\xi$-closed.

2.7 Example Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$ Then $A = \{a\}$ is $\alpha$-closed as well as $\xi$-closed but not closed.

2.8 Example Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c, d\}, X\}$. Then $A = \{a, b, c\}$ is $\xi$-closed. But it is neither $\alpha$-closed nor closed.

2.9 Lemma. A subset $A$ of a space $X$ is $g\xi$-open in $X$ if and only if $F \subseteq \xi\text{Int}(A)$ whenever $F \subseteq A$ and $F$ is closed in $X$.

3. Generalized $\xi$-closed functions

3.1 Definition. A function $f : X \to Y$ is said to be $\xi$-closed [2] if for each closed set $F$ of $X$, $f(F)$ is $\xi$-closed in $Y$.

3.2 Definition. A function $f : X \to Y$ is said to be

(i) generalized $\xi$-closed (briefly $g\xi$-closed) if for each closed set $F$ of $X$, $f(F)$ is $g\xi$-closed in $Y$.

(ii) $\xi$-generalized $\xi$-closed (briefly $\xi$-$g\xi$-closed) if for each $\xi$-closed set $F$ of $X$, $f(F)$ is $g\xi$-closed in $Y$.

3.3 Remark. Every closed function is $\xi$-closed but not conversely. Also, every $\xi$-closed function is $g\xi$-closed because every $\xi$-closed set is $g\xi$-closed. It is obvious that both $\xi$-closedness and $g\xi$-closedness imply $\xi$-closedness.

3.4 Theorem. A surjective function $f : X \to Y$ is $g\xi$-closed (resp. $\xi$-$g\xi$-closed) if and only if for each subset $B$ of $Y$ and each open (resp. $\xi$-open) set $U$ of $X$ containing $f^{-1}(B)$, there exists a $g\xi$-open set $V$ of $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. Suppose that $f$ is $g\xi$-closed (resp. $\xi$-$g\xi$-closed). Let $B$ be any subset of $Y$ and $U$ be open (resp. $\xi$-open) set of $X$ containing $f^{-1}(B)$. Put $V = Y - f(B)$. Then the complement $V^c$ of $V$ in $Y$ is $V^c = Y - V = f(B)$, since $X - U$ is closed in $X$ and $f$ is $g\xi$-closed, $f(X - U) = V^c$ is $g\xi$-closed. Therefore, $V = g\xi$-open in $Y$. It is easy to see that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Conversely, let $F$ be a closed (resp. $\xi$-closed) set of $X$. Put $B = Y - f(F)$, then we have $f^{-1}(B) \subseteq X - F$ and $X - F$ is open (resp. $\xi$-open) in $X$. Then by assumption, there exists a $g\xi$-open set $V$ of $Y$ such that $B = Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Now $f^{-1}(V) \subseteq X - F$ implies $V \subseteq Y - f(F) = B$. Also, $B \subseteq V$ and so $B = V$. Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is $g\xi$-closed in $Y$. This shows that $f$ is $g\xi$-closed (resp. $\xi$-$g\xi$-closed).

3.5 Remark. We can prove the necessity part of the above theorem by replacing each set to closed set in the form of the proposition given below.

3.6 Proposition. If a surjective function $f : X \to Y$ is $g\xi$-closed (resp. $\xi$-$g\xi$-closed) then for a closed set $F$ of $Y$ and for any open (resp. $\xi$-open) set $U$ of $X$ containing $f^{-1}(F)$, there exists a $\xi$-open set $V$ of $Y$ such that $F \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof. By Theorem 3.4, there exists a $g\xi$-open set $W$ of $Y$ such that $F \subseteq W$ and $f^{-1}(W) \subseteq U$. Since $F$ is closed, by Lemma 2.9 we have $F \subseteq \xi\text{Int}(W)$. Put $V = \xi\text{Int}(W)$. Then $V \subseteq \xi\text{Int}(Y)$, $F \subseteq V$ and $f^{-1}(V) \subseteq U$.

3.7 Proposition. If $f : X \to Y$ is continuous $\xi$-$g\xi$-closed and $A$ is $g\xi$-closed in $X$, then $f(A)$ is $g\xi$-closed in $Y$.

Proof. Let $V$ be a open set of $Y$ containing $f(A)$. Then $A \subseteq f^{-1}(V)$. Since $f$ is continuous, $f^{-1}(V)$ is open in $X$. Since $A$ is $g\xi$-closed in $X$, by a definition, we get $\xi\text{Cl}(A) \subseteq f^{-1}(V)$ and hence $f(\xi\text{Cl}(A)) \subseteq V$. Since $f$ is $g\xi$-$g\xi$-closed and $\xi\text{Cl}(A)$ is $\xi$-closed in $X$, $f(\xi\text{Cl}(A))$ is $g\xi$-closed in $Y$ and hence we have $\xi\text{Cl}(f(\xi\text{Cl}(A))) \subseteq V$. By definition of the $\xi$-closure of a set, $A \subseteq \xi\text{Cl}(A)$ which implies $f(A) \subseteq f(\xi\text{Cl}(A))$ and using Lemma 2.2, $\xi\text{Cl}(f(A)) \subseteq \xi\text{Cl}(f(\xi\text{Cl}(A))) \subseteq U$. That is $\xi\text{Cl}(f(A)) \subseteq U$. This shows that $f(A)$ is $g\xi$-closed in $Y$.

3.8 Definition. A function $f : X \to Y$ is said to be $\xi$-irresolute [2] if for each $V \subseteq \xi\text{O}(Y)$, $f^{-1}(V) \subseteq \xi\text{O}(X)$.

3.9 Proposition. If $f : X \to Y$ is an open $\xi$-irresolute bijection and $B$ is $g\xi$-closed in $Y$, then $f^{-1}(B)$ is $g\xi$-closed in $X$.

Proof. Let $U$ be an open set of $X$ containing $f^{-1}(B)$. Then $B \subseteq f(U)$ and $f(U)$ is open in $Y$. Since $B$ is $g\xi$-closed in $Y$, $\xi\text{Cl}(B) \subseteq f(U)$ and hence we have $f^{-1}(\xi\text{Cl}(B)) \subseteq U$. Since $f$ is $\xi$-irresolute, $f^{-1}(\xi\text{Cl}(B))$ is $\xi$-closed in $X$ (Theorem 2.1 (ii) and (iv)), we have $\xi\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(\xi\text{Cl}(B)) \subseteq U$. This shows that $f^{-1}(B)$ is $g\xi$-closed in $X$.

3.10 Theorem. Let $f : X \to Y$ and $h : Y \to Z$ be the two functions, then

(i) If $h \circ f : X \to Z$ is $g\xi$-closed and if $f : X \to Y$ is a continuous surjection, then $h : X \to Z$ is $g\xi$-closed.

(ii) If $f : X \to Y$ is $g\xi$-closed with $h : Y \to Z$ is continuous and $\xi$-$g\xi$-closed, then $h \circ f : X \to Z$ is $g\xi$-closed.

(iii) If $f : X \to Y$ is closed and $h : Y \to Z$ is $g\xi$-closed, then $h \circ f : X \to Z$ is $g\xi$-closed.

Proof.

(i) Let $F$ be a closed set of $X$. Then $f^{-1}(F)$ is closed in $X$ since $f$ is continuous. By hypothesis (hof) ($f^{-1}(F)$) is $g\xi$-closed in $Z$. Hence $h$ is $g\xi$-closed.

(ii) The proof follows from the Proposition 3.7.

(iii) The proof is obvious from definitions.
4. ξ-Normal spaces

4.1 Definition. A space X is said to be ξ-normal (resp. α-normal [1], p-normal [8, 9], β-normal [4]) if for any pair of disjoint closed sets A, B of X, there exist disjoint ξ-open (resp. α-open, p-open, β-open) sets U and V such that A ⊂ U and B ⊂ V.

By the definitions stated above, we have the following diagram:

normality ⇒ α-normality ⇒ p-normality ⇒ β-normality

ξ-normality

Where none of the implications is reversible as can be seen from the following examples:

4.2 Example. Let X = {a, b, c, d} and = {φ, {b}, {d}, {b, d}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Taking β-open sets, U = {a, b} and V = {c, d} such that A ⊂ U and B ⊂ V. Hence the space X is β-normal. But the space X is neither p-normal nor α-normal, since the sets U and V are neither open nor α-open.

4.3 Example. Let X = {a, b, c, d} and τ = ∅, {b, d}, {a, b, d}, {b, c, d}, X. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Taking β-open sets, U = {a, b} and V = {c, d} such that A ⊂ U and B ⊂ V. Hence the space X is p-normal as well as β-normal, since every p-open sets are β-open. But the space X is neither normal nor α-normal, since the sets U and V are neither open nor α-open.

4.4 Theorem. The following properties are equivalent for a space X:

(a) X is ξ-normal.
(b) For each pair of disjoint closed sets A, B of X, there exist disjoint gξ-open sets U and V such that A ⊂ U and B ⊂ V.
(c) For each closed set A and any open set V containing A, there exists a gξ-open set U such that A ⊂ U ⊂ ξCl(U) ⊂ V.
(d) For each closed set A and any open set B containing A, there exists a gξ-open set U such that A ⊂ U ⊂ ξCl(U) ⊂ Int(B).
(e) For each closed set A and any open set B containing A, there exists a ξ-open set S such that A ⊂ S ⊂ ξCl(A) ⊂ Int(B).
(f) For each closed set A and any open set B containing A, there exists a ξ-open set S such that C1(A) ⊂ S ⊂ ξCl(A) ⊂ B.
(g) For each closed set A and any open set B containing A, there exists a gξ-open set S such that C1(A) ⊂ S ⊂ ξCl(A) ⊂ B.

Proof. (a) ⇒ (b). This proof is obvious since every ξ-open set is gξ-open.

(b) ⇒ (c). Let A be a closed set and let V be an open set containing A. Since A and X – V are disjoint closed sets of X, there exist gξ-open sets U and W of X such that A ⊂ U and X – V ⊂ W and U ∩ W = φ. By Lemma 2.9, we get X - V ⊂ ξInt(W). Since U ∩ ξInt(W) = φ, we have ξCl(U) ∩ ξInt(W) = φ and hence ξCl(U) ⊂ X - ξInt(W) ⊂ V. Therefore, we obtain A ⊂ U ⊂ ξCl(U).

(c) ⇒ (a). Let A and B be the disjoint closed sets of X. Since X – B is an open set containing A, there exists a gξ-open set G such that A ⊂ G ⊂ ξCl(G) ⊂ X – B. By Lemma 2.9, we have A ⊂ ξInt(G). Put U = ξInt(G) and V = X – ξCl(G). Then U and V are disjoint ξ-open sets such that A ⊂ U and B ⊂ V. Therefore X is ξ-normal.

Since every ξ-open set is gξ-open and every closed (resp. open) set is g-closed (resp. g-open), it is obvious that (d) ⇒ (e) ⇒ (c) and (f) ⇒ (g) ⇒ (c).

(c) ⇒ (e). Let A be a closed set of X and let B be a g-open set such that A ⊂ B. Since B is g-open and A is closed, A ⊂ Int(B) by Lemma 2.9. Therefore by (c), there exists a gξ-open set U such that A ⊂ U ⊂ ξCl(U) ⊂ ξInt(B).

(e) ⇒ (d). Let A be a closed set of X and let B be a g-open set such that A ⊂ B. Then there exists a gξ-open set G such that A ⊂ G ⊂ ξCl(G) ⊂ Int(B) by Lemma 2.9. Since G is gξ-open, A ⊂ ξInt(G). Put U = ξInt(G), then U is ξ-open and A ⊂ U ⊂ ξCl(U) ⊂ Int(B).

(d) ⇒ (g). Let A be a g-closed set of X and let B be an open set such that A ⊂ B. Then Cl(A) ⊂ B. By (c), there exists a gξ-open set U. Such that Cl(A) ⊂ U ⊂ ξCl(U) ⊂ B.

(g) ⇒ (f). Let A be a g-closed set of X and let B be an open set such that A ⊂ B. Then there exist a gξ-open set G such that Cl(A) ⊂ G ⊂ ξCl(G) ⊂ Int(B). Since G is gξ-open and the closed set Cl(A) ⊂ G, we have Cl(A) ⊂ ξInt(G) by Lemma 2.9. Put U = ξInt(G). Then, U is ξ-open and Cl(A) ⊂ U ⊂ ξCl(U) ⊂ B.

4.5 Theorem. If f: X → Y is continuous gξ-closed surjection and X is normal, then Y is ξ-normal.

Proof. Let A and B be the disjoint closed sets of Y. Then f⁻¹(A) and f⁻¹(B) are disjoint closed sets of X since f is continuous. Since X is normal, there exists disjoint open sets U and V such that f⁻¹(A) ⊂ U and f⁻¹(B) ⊂ V. By Proposition 3.6, there exist gξ-open sets G and H of Y such that A ⊂ G, B ⊂ H and f⁻¹(G) ⊂ U and f⁻¹(H) ⊂ V. Then we have f⁻¹(G) ∩ f⁻¹(H) = φ and hence G ∩ H = φ. It follows from Theorem 4.4 that Y is ξ-normal.

4.6 Theorem. If f: X → Y is continuous ξ-gξ-closed surjection and X is ξ-normal, then Y is ξ-normal.
Proof. Let A and B the disjoint closed sets of Y. Then \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint closed sets of X. Since X is \( \xi \)-normal, there exist disjoint \( \xi \)-open sets U and V such that \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). Since f is \( \xi \)-\( \xi \)-closed, by Proposition 3.6, there exist \( \xi \)-open sets G and H of Y such that A \( \subset \) G, B \( \subset \) H, \( f^{-1}(G) \subset U \) and \( f^{-1}(H) \subset V \). Since U and V are disjoint, we have \( G \cap H = \phi \). This shows that Y is \( \xi \)-normal.

5. \( \xi \)-Regular Spaces

5.1 Definition. A space X is said to be \( \xi \)-regular [10] (resp. \( \alpha \)-regular [11]) if for each closed set F of X, and each point \( x \in X - F \), there exist disjoint \( \xi \)-open (resp. \( \alpha \)-open) set U, V such that \( U \cap V = \phi \).

5.2 Remark. It is obvious that every \( \alpha \)-regular space is \( \xi \)-regular but not conversely.

5.3 Lemma. The following properties are equivalent for a space X:

(a) X is \( \xi \)-regular.

(b) For each \( x \in X \) and each open set U of X containing x, there exists \( V \in \xi (X) \) such that \( x \in V \subset \xi (V) \subset U \).

(c) For each closed set F of X, \( \cap \{ \xi (V) / F \subset V \in \xi (O(X)) \} = F \).

(d) For each subset A of X and each open set U of X such that \( A \cap U = \phi \), there exists \( V \in \xi (O(X)) \) such that \( A \cap V \neq \phi \) and \( \xi (V) \subset U \).

(e) For each non-empty subset A of X and each closed subset F of X such that \( A \cap F = \phi \), there exist \( V, W \in \xi (O(X)) \) such that \( A \cap V \neq \phi \), \( F \subset W \) and \( V \cap W = \phi \).

Proof.

\((a) \Rightarrow (b)\). Let \( U \subset X \) be an open set containing \( x \), then \( x \in U \) is closed in X and \( x \notin U - U \). By \( (a) \), there exist \( W \subset V \in \xi (O(X)) \) such that \( x \in V, X - U \subset W \) and \( V \cap W = \phi \). By Lemma 2.2, we have \( \xi (Cl(V)) \cap W = \phi \) and hence \( x \in V \subset \xi (Cl(V)) \subset U \).

\((b) \Rightarrow (c)\). Let \( F \subset X \). If \( F \subset V \), then by Lemma 2.2 (iii), \( \xi (F) \subset \xi (Cl(V)) \) which gives \( F \subset \xi (Cl(V)) \) as \( F \subset \xi (F) \). Therefore, \( \cap \{ \xi (Cl(V)) / F \subset V \in \xi (O(X)) \} \supset F \).

Conversely, let \( x \notin F \). Then \( X - F \) is an open set containing x. By \( (b) \), there exists \( U \in \xi (O(X)) \) such that \( x \in U \subset \xi (Cl(U)) \subset X - F \). Put \( V = X - \xi (Cl(U)) \). By Lemma 2.2, \( F \subset V \in \xi (O(X)) \) and \( x \notin \xi (Cl(V)) \). This implies \( \cap \{ \xi (Cl(V)) / F \subset V \in \xi (O(X)) \} \supset F \).

\((c) \Rightarrow (d)\). Let A be a subset of X and let \( U \in X \) be open in X such that \( A \cap U = \phi \). Let \( x \in A \cap U \), then \( x \in U \) is a closed set not containing x. By \( (c) \), there exists W of \( \xi (O(X)) \) such that \( x \in U \subset W \) and \( x \notin \xi (Cl(W)) \). Put \( V = X - \xi (Cl(W)) \). Then \( V \subset X - W \). Also, \( x \in V \cap A \). By using

Lemma 2.2, we obtain \( V \in \xi (O(X)) \) and \( \xi (Cl(V)) \subset \xi (Cl(X - W)) = X - W \subset U \).

\((d) \Rightarrow (e)\). Let \( A \subset X \) be a closed set and let \( F \subset A \) be a closed set in X such that \( A \cap F = \phi \), where \( A \neq \phi \). Since \( X - F \) is open in X and \( A \neq \phi \), by \((d)\), there exists \( V \in \xi (O(X)) \) such that \( A \cap V \neq \phi \) and \( \xi (Cl(V)) \subset X - F \). Put \( W = X - \xi (Cl(V)) \). Then \( F \subset W \). Also, \( V \cap W = \phi \). By Lemma 2.2, \( W \in \xi (O(X)) \).

\((e) \Rightarrow (a)\). This is obvious.

5.4 Theorem. The following properties are equivalent for a space X:

(a) X is \( \xi \)-regular.

(b) For each closed set \( F \) and each point \( x \in X - F \), there exists \( U \in \xi (O(X)) \) and a \( g\xi \)-open set \( V \) such that \( x \in U \) and \( F \subset V \) and \( U \cap V = \phi \).

(c) For each subset \( A \) of X and each closed set \( F \) such that \( A \cap F = \phi \), there exist \( U \in \xi (O(X)) \) and a \( g\xi \)-open set \( V \) such that \( A \cap U \neq \phi \), \( F \subset V \) and \( U \cap V = \phi \).

(d) For each closed set \( F \) of X, \( \cap \{ \xi (Cl(V)) / F \subset V \in \xi (O(X)) \} \).

Proof.

\((a) \Rightarrow (b)\). The proof is obvious since every \( \xi \)-open set is \( g\xi \)-open.

\((b) \Rightarrow (c)\). Let A be a subset of X and let \( F \) be a closed set in X such that \( A \cap F = \phi \). For a point \( x \in A \), \( x \in X - F \) and hence there exists \( U \in \xi (O(X)) \) and a \( g\xi \)-open set \( V \) such that \( x \in U \) and \( F \subset V \) and \( U \cap V = \phi \). Also \( x \in A \), \( x \in U \) implies \( x \in A \subset U \). So \( A \cap U \neq \phi \).

\((c) \Rightarrow (a)\). Let \( F \) be a closed set and let \( x \in X - F \). Then, \( \cap \{ \xi (Cl(V)) / F \subset V \in \xi (O(X)) \} \).

\((d) \Rightarrow (a)\). Let \( F \) be a closed set of X and \( x \in X - F \). By \((d)\), there exists a \( g\xi \)-open set \( W \) of X such that \( F \subset W \) and \( x \in X - \xi (Cl(W)) \). Since F is closed, \( F \subset \xi (Int(W)) \) by Lemma 2.9, we have \( F \subset V, V \in \xi (O(X)) \) and \( U \cap V = \phi \). Therefore X is \( \xi \)-regular.

5.5 Definition. A function \( f : X \to Y \) is said to be \( \xi \)-open [2] if for each open set \( U \) of X, \( f(U) \in \xi (O(Y)) \).
5.6 **Theorem.** If \(f: X \to Y\) is a continuous \(\xi\)-open \(g\xi\)-closed surjection and \(X\) is regular, then \(Y\) is \(\xi\)-regular.

**Proof.** Let \(y \in Y\) and let \(V\) be an open set of \(Y\) containing \(y\). Let \(x\) be a point of \(X\) such that \(y = f(x)\). By the regularity of \(X\), there exists an open set \(U\) of \(X\) such that \(x \in U \subset C1(U) \subset f^{-1}(V)\). We have \(y \in f(U) \subset f(C1(U)) \subset V\), since \(f\) is \(\xi\)-open and \(g\xi\)-closed, \(f(U) \in \xi\mathcal{O}(Y)\) and \(f(C1(U)) = g\xi\)-closed in \(Y\). So, we obtain, \(y \in f(U) \subset \xi C1(f(U)) \subset \xi(C1(f(U))) \subset V\). It follows from **Lemma 5.4** that \(Y\) is \(\xi\)-regular.

5.7 **Definition.** A function \(f: X \to Y\) is said to be **pre \(\xi\)-open** if for each \(\xi\)-open set \(U\) of \(X\), \(f(U) \in \xi\mathcal{O}(Y)\).

5.8 **Theorem.** If \(f: X \to Y\) is a continuous pre \(\xi\)-open \(g\xi\)-closed surjection and \(X\) is \(\xi\)-regular, then \(Y\) is \(\xi\)-regular.

**Proof.** Let \(F\) be any closed set of \(Y\) and \(y \in Y - F\). Then \(f^{-1}(Y) \cap f^{-1}(F) = \phi\) and \(f^{-1}(F)\) is closed in \(X\). Since \(X\) is \(\xi\)-regular, for a point \(x \in f^{-1}(y)\), there exist \(U, V \in \xi\mathcal{O}(X)\) such that \(x \in U, f^{-1}(F) \subset V\) and \(U \cap V = \phi\). Since \(F\) is closed in \(Y\), by **Proposition 3.6**, there exists \(W \in \xi\mathcal{O}(Y)\) such that \(F \subset W\) and \(f^{-1}(W) \subset V\). Since \(f\) is pre \(\xi\)-open, we have \(y = f(x) \in f(U)\) and \(f(U) \in \xi\mathcal{O}(Y)\). Since \(U \cap V = \phi\), \(f^{-1}(W) \cap U = \phi\) and hence \(W \cap f(U) = \phi\). This shows that \(Y\) is \(\xi\)-regular.

6. Conclusion

We introduced a weaker version of normality called \(\xi\)-normality in topological spaces. We gave some characterizations and preservation theorems of \(\xi\)-normal and \(\xi\)-regular spaces. Some counterexamples were given and some basic properties were presented. The relationships among normal, \(\alpha\)-normal, \(p\)-normal, \(\beta\)-normal, and \(\xi\)-normal are investigated.

**References**


