

Integral Representations of Analytical Functions Associated with Conic Regions

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Abstract: In this paper, we define generalized subclasses of k -uniformly Janowski starlike and k -uniformly Janowski convex functions associated with m -symmetric points. Interesting integral representations for functions belonging to this class are investigated.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad (1.1)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Furthermore S represents class of all functions in A which are univalent in U . Generalizing the well known class of star like functions Sakaguchi [6] introduced a class S_s^* of functions star like with respect to symmetric points, consisting of functions $f \in S$ satisfying the inequality

$$\Re \left(\frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in U. \quad (1.2)$$

Analogously Das and Singh [5] in 1977 extend the results of Sakaguchi to other class in U namely convex functions with respect to symmetric points. Let C_s denote the class of convex functions with respect to symmetric points and satisfying the condition

$$\Re \left(\frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in U$$

We note that C_s and S_s^* are connected by the well-known Alexander relation. i.e., $f \in C_s$ if and only if $zf'(z) \in S_s^*$. Chand and Singh [11] introduced class S_s^* of functions starlike with respect to m -symmetric points, which consists of functions $f \in S$, satisfying the inequality

$$\Re \left(\frac{zf'(z)}{f_m(z)} \right) > 0, \quad z \in U. \quad (1.3)$$

where

$$f_m(z) = \frac{1}{m} \sum_{\mu=0}^{m-1} \varepsilon^{-\mu} f(\varepsilon^\mu z), \quad (\varepsilon^\mu = 1, \quad m \in \mathbb{N}) \quad (1.4)$$

From (1.4) we can write

$$\begin{aligned} f_m(z) &= \frac{1}{m} \sum_{\mu=0}^{m-1} \varepsilon^{-\mu} f(\varepsilon^\mu z) = \frac{1}{m} \sum_{\mu=0}^{m-1} \varepsilon^{-\mu} \left(\varepsilon^\mu z + \sum_{n=2}^{\infty} a_n (\varepsilon^\mu z)^n \right) \\ &= z + \sum_{n=2}^{\infty} a_n b_n z^n \end{aligned} \quad (1.5)$$

Where

$$b_n = \frac{1}{m} \sum_{\mu=0}^{m-1} \varepsilon^{(n-1)\mu} = \begin{cases} 1, & n = lm + 1 \\ 0, & n \neq lm + 1, \end{cases} \quad (1.6)$$

Where $l, m \in \mathbb{N}, n \geq 2, \varepsilon^m = 1$.

The following equations follow directly from the above definition [1],

$$f_m(\varepsilon^\mu z) = \varepsilon^\mu f_m(z) \quad (1.7)$$

$$f'_m(\varepsilon^\mu z) = f'_m(z) = \frac{1}{m} \sum_{\mu=0}^{m-1} f'(\varepsilon^\mu z), \quad z \in U \quad (1.8)$$

The convolution or Hadamard product of two power series $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$

is defined as the power series $(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m$.

Now we define the concept of subordination.

Definition 1.1. [8] Let Ω be the set of analytic functions ω such that $\omega(0) = 0, |\omega(z)| < 1, z \in U$. For any two analytic functions f and g defined in the unit disc U , we say that f is subordinate to g in U if there exists a function $\omega \in \Omega$ such that

$$f(z) = g(\omega(z)) \quad (z \in U). \quad (1.9)$$

If f is subordinate to g on U we denote this by $f \prec g$.

Definition 1.2. Let P denote the class of analytic functions of the form

$$p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m, \quad \text{defined in } U \text{ satisfying } p(0) = 1 \text{ and } \Re\{p(z)\} > 0 \quad (z \in U).$$

The class P can be completely characterized in terms of subordination. In fact the linear fractional transformation

$$\omega = g(z) = \frac{1+z}{1-z} \quad (z \in U)$$

maps the disc U univalently onto the right half plane $\Re\{\omega\} > 0$ and we have $g(0) = 1$. It follows from the definition that a function p is in P if and only if

$$p(z) \prec \frac{1+z}{1-z}. \quad \text{Hence any function } p \in P \text{ has the representation [9],}$$

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} \quad (\omega \in \Omega, z \in U).$$

This representation for functions with positive real part in terms of analytic functions defined in U satisfying the conditions of Schwarz lemma [9] motivated Janowski [10] to define a new class $P(A, B)$.

Definition 1.3. [10] Let $P(A,B)(-1 \leq B < A \leq 1)$, denote the class of analytic functions p defined in U with the representation

$$p(z) = \frac{1+A\omega(z)}{1-B\omega(z)} \quad (\omega \in \Omega, z \in U).$$

Further $P(1, -1) = P$.

Kanas and Wisniowska [3, 4] introduced and studied the class k -UCV of k -uniformly convex functions and the corresponding class k -ST of k -star like functions. These classes were defined subject to the conic domain $\Omega_k, k \geq 0$ given by

$$p_{k,\sigma} = \begin{cases} \frac{1+(1-2\sigma)z}{1-z}, & k = 0, \\ 1 + \frac{2(1-\sigma)}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1, \\ 1 + \frac{2(1-\sigma)}{1-k^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos k \right) \operatorname{arctanh} \sqrt{z} \right], & 0 < k < 1, \\ 1 + \frac{(1-\sigma)}{k^2-1} \sin \left[\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} dx \right] + \frac{1}{k^2-1}, & k > 1, \end{cases} \quad (1.11)$$

Where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tx}}, t \in (0, 1), z \in U$ and t is chosen such that $k = \cosh\left(\frac{\pi R(t)}{4R(t)}\right)$, with $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral $R(t)$.

Let $P_{k,\sigma}$ denote the class of all functions p which are analytic in U with $p(0) = 1$ and $p(z) \prec_{p_{k,\sigma}}(z)$ for $z \in U$. Clearly, it can be seen that $P_{k,\sigma}(z) \subset P$.

Definition 1.4. [5] A function p is said to be in k - $P[A, B]$, if and only if,

$$p(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \quad k \geq 0, \quad (1.12)$$

where $p_k(z)$ is defined by (1.11) and $-1 \leq B < A \leq 1$.

Geometrically, the function $p \in k$ - $P[A, B]$ takes all values from the domain $\Omega_k[A, B], -1 \leq B < A \leq 1, k \geq 0$ which is defined as

$$\Omega_k[A, B] = \left\{ \omega: R \left(\frac{(B+1)\omega(z)-(A-1)}{(B+1)\omega(z)-(A+1)} \right) > k \left| \frac{(B-1)\omega(z)-(A-1)}{(B+1)\omega(z)-(A+1)} - 1 \right| \right\} \quad (1.13)$$

$$\text{Or equivalently, } \Omega_k[A, B] = \left\{ u + iv: [(B^2 - 1)(u^2 + v^2) - 2(AB - 1)u + (A^2 - 1)]^2 > k^2 [(-2(B + 1)(u^2 + v^2) + 2(A + B + 2)u - 2(A + 1))^2 + 4(A - B)^2 v^2] \right\}.$$

The domain $\Omega_k[A, B]$ retains the conic domain Ω_k inside the circular region defined by $\Omega[A, B]$. The impact of $\Omega[A, B]$ on the conic domain Ω_k changes the original shape of the conic regions. The ends of hyperbola and parabola get closer to each other but never meet anywhere and the ellipse gets the oval shape. When $A \rightarrow 1, B \rightarrow -1$, the radius of the circular disk defined by $\Omega[A, B]$ tends to infinity, consequently the arms of hyperbola and parabola expand and the oval turns into ellipse. We see that $\Omega_k[1, -1] = \Omega_k$, the conic domain defined by Kanas and Wisniowska [3,4].

Motivated essentially by the recent paper of Noor and Malik [7], we define some classes of analytic functions associated with conic domains as follows:

$$(1.10) \quad \Omega_k = \{u + iv : u > k\sqrt{(u-1)^2 + v^2}\}.$$

This domain represents the right half plane for $k = 0$, hyperbola for $0 < k < 1$, a parabola for $k = 1$ and ellipse for $k > 1$.

They also extended this domain to $\Omega_{k,\sigma}$ defined by

$$\Omega_{k,\sigma} = (1 - \sigma)\Omega_k + \sigma \quad (0 \leq \sigma < 1).$$

The function $p_{k,\sigma}$, with $p_{k,\sigma}(0) = 1, p'_{k,\sigma} > 0$ plays the role of extremal for the conic domain $\Omega_{k,\sigma}$ and is given by

Definition 1.5. A function p is said to be in the class k - $P[A, B, \sigma]$, if and only if

$$p(z) \prec \frac{(A+1)p_{k,\sigma}(z)-(A-1)}{(B+1)p_{k,\sigma}(z)-(B-1)}, \quad k \geq 0,$$

where $p_{k,\sigma}(z)$ is defined by (1.11), $0 \leq \sigma < 1$ and $-1 \leq B < A \leq 1$.

Definition 1.6. A function $f \in A$ is said to be in the class k - $ST_s^{(m)}[A, B, \sigma]$,

$-1 \leq B < A \leq 1, k \geq 0$, if and only if

$$R \left(\frac{(B-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_m(z)}-(A+1)} \right) > k \left| \frac{(B-1)\frac{zf'(z)}{f_m(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f_m(z)}-(A+1)} - 1 \right|$$

Or equivalently

$$\frac{zf'(z)}{f_m(z)} \in k - P[A, B, \sigma] \quad (1.14)$$

where f_m is defined by (1:5).

Definition 1.7. A function $f \in A$ is said to be in the class k - $UCV_s^{(m)}[A, B, \sigma]$,

$-1 \leq B < A \leq 1, k \geq 0$, if and only if

$$R \left(\frac{(B-1)\left(\frac{zf'(z)}{f_m(z)}\right)'-(A-1)}{(B+1)\left(\frac{zf'(z)}{f_m(z)}\right)'-(A+1)} \right) > k \left| \frac{(B-1)\left(\frac{zf'(z)}{f_m(z)}\right)'-(A-1)}{(B+1)\left(\frac{zf'(z)}{f_m(z)}\right)'-(A+1)} - 1 \right|$$

Or equivalently

$$\left(\frac{zf'(z)}{f_m(z)}\right)' \in k - P[A, B, \sigma] \quad (1.14)$$

where f_m is defined by (1:5).

As special cases, we get the results of Nasir Khan [12], Noor and Malik [7], Kwon and Sim [1], Kanas and Wisniowska [3], Shams, Kulkarni and Jahangiri [2], Janowski [13], Kanas and Wisniowska [4].

2. Integral Representation

We give two meaningful conclusions about the classes k - $ST_s^{(m)}[A, B, \sigma]$ and k - $UCV_s^{(m)}[A, B, \sigma]$.

Theorem 2.1. Let $f \in k-ST_s^{(m)}[A, B, \sigma]$. Then $f_m \in k - ST[A, B, \sigma] \subseteq k - ST \subseteq S$.

Proof: If $f \in k-ST_s^{(m)}[A, B, \sigma]$, we can obtain

$$\frac{zf'(z)}{f_m(z)} < \frac{(A+1)p_{k,\sigma}(z)-(A-1)}{(B+1)p_{k,\sigma}(z)-(B-1)}, \quad (z \in U) \quad (2.1)$$

Substituting z by $\epsilon^\mu z$ respectively ($\mu = 0, 1, 2, \dots, m-1$).

$$\frac{\epsilon^\mu zf'(\epsilon^\mu z)}{f_m(\epsilon^\mu z)} < \frac{(A+1)p_{k,\sigma}(\epsilon^\mu z)-(A-1)}{(B+1)p_{k,\sigma}(\epsilon^\mu z)-(B-1)} < \frac{(A+1)p_{k,\sigma}(z)-(A-1)}{(B+1)p_{k,\sigma}(z)-(B-1)}, \quad (z \in U) \quad (2.2)$$

By definition of f_m and $\epsilon = \exp\left(\frac{2\pi i}{m}\right)$, we know that $\epsilon^{-\mu} f_m(\epsilon^\mu z) = f_m(z)$. Then equation (2.2), becomes

$$\frac{zf_m'(z)}{f_m(z)} < \frac{(A+1)p_{k,\sigma}(z)-(A-1)}{(B+1)p_{k,\sigma}(z)-(B-1)}, \quad (z \in U) \quad (2.3)$$

Let ($\mu = 0, 1, 2, \dots, m-1$) in (2.3), respectively and sum them to get

$$\frac{zf_m'(z)}{f_m(z)} < \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{zf_m'(\epsilon^\mu z)}{f_m(z)} < \frac{(A+1)p_{k,\sigma}(z)-(A-1)}{(B+1)p_{k,\sigma}(z)-(B-1)}, \quad (z \in U)$$

Thus $f_m \in k - ST[A, B, \sigma] \subseteq S$.

For $\sigma = 0$ we obtain the following Corollary, which is comparable to the result obtained by Nasir Khan [12].

Corollary 2.2: Let $f \in k-ST_s^{(m)}[A, B]$. Then $f_m \in k - ST[A, B] \subseteq k - ST \subseteq S$.

Putting $\sigma = 0$ and $k = 0$ we can obtain below result, which is comparable to the result obtained by Kwon and Sim[1].

Corollary 2.3: Let $f \in ST_s^{(m)}[A, B]$. Then $f_m \in ST[A, B] \subseteq k - ST \subseteq S$.

Similar to the proof of Theorem 2.1, we can prove the following Theorem.

Theorem 2.4. Let $f \in k-UCV_s^{(m)}[A, B, \sigma]$. Then $f_m \in k - UCV[A, B, \sigma] \subseteq S$.

Now we give the integral representations of the functions belonging to the classes

$k-ST_s^{(m)}[A, B, \sigma]$ and $k-UCV_s^{(m)}[A, B, \sigma]$.

Theorem 2.5. Let $f \in k-ST_s^{(m)}[A, B, \sigma]$. Then

$$f_m(z) = z \left(\exp(A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\epsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)-1)}{t(B+1)p_{k,\sigma}(\omega(t))-(B-1)} dt \right), \quad (2.4)$$

Where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Proof: Let $f \in ST_s^{(m)}[A, B, \sigma]$, from definition of the subordination, we can have

$$\frac{zf'(z)}{f_m(z)} < \frac{(A+1)p_{k,\sigma}(\omega(z))-(A-1)}{(B+1)p_{k,\sigma}(\omega(z))-(B-1)}, \quad (z \in U) \quad (2.5)$$

Where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Substituting z by $\epsilon^\mu z$ respectively ($\mu = 0, 1, 2, \dots, m-1$), we have

$$\frac{\epsilon^\mu zf'(\epsilon^\mu z)}{f_m(\epsilon^\mu z)} = \frac{(A+1)p_{k,\sigma}(\omega(\epsilon^\mu z))-(A-1)}{(B+1)p_{k,\sigma}(\omega(\epsilon^\mu z))-(B-1)}, \quad (z \in U) \quad (2.6)$$

Using the equations (1.7) and (1.8) we have

$$\frac{zf'(z)}{f_m(z)} = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{(A+1)p_{k,\sigma}(\omega(\epsilon^\mu z))-(A-1)}{(B+1)p_{k,\sigma}(\omega(\epsilon^\mu z))-(B-1)}, \quad (z \in U) \quad (2.7)$$

Or equivalently

$$\frac{f_m'(z)}{f_m(z)} - \frac{1}{z} = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{(A-B)(p_{k,\sigma}(\omega(\epsilon^\mu z)) - 1)}{z((B+1)p_{k,\sigma}(\omega(\epsilon^\mu z)) - (B-1))}, \quad (z \in U) \quad (2.8)$$

Integrating equality (2.8), we have

$$\log \frac{f_m(z)}{z} = (A-B) \frac{1}{m} \left(\sum_{\mu=0}^{m-1} \int_0^{\epsilon^\mu z} \frac{p_{k,\sigma}(\omega(\epsilon^\mu \zeta)) - 1}{\zeta(B+1)p_{k,\sigma}(\omega(\epsilon^\mu \zeta)) - (B-1)} d\zeta \right), \quad (2.9)$$

$$= (A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\epsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)) - 1}{t(B+1)p_{k,\sigma}(\omega(t)) - (B-1)} dt$$

Therefore arranging equality (2.9) for $f_m(z)$ we can obtain

$$f_m(z) = z \left(\exp(A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\epsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)) - 1}{t(B+1)p_{k,\sigma}(\omega(t)) - (B-1)} dt \right)$$

and so the proof of the Theorem 2.5 is complete.

For $\sigma = 0$ we obtain the following Corollary, which is comparable to the result obtained by Nasir Khan [12].

Corollary 2.6. Let $f \in ST_s^{(m)}[A, B]$. Then

$$f_m(z) = z \left(\exp(A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\epsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)) - 1}{t(B+1)p_{k,\sigma}(\omega(t)) - (B-1)} dt \right) \quad (2.10)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Putting $m=1$ in Theorem 2.5 we obtain the following corollary.

Corollary 2.7. Let $f \in k-ST[A, B, \sigma]$. Then

$$f(z) = z \left(\exp(A-B) \int_0^z \frac{p_{k,\sigma}(\omega(t)) - 1}{t(B+1)p_{k,\sigma}(\omega(t)) - (B-1)} dt \right), \quad (2.11)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Putting $\sigma = 0$ and $k=0$ in Theorem 2.5 we obtain the following corollary. Which is comparable to the result obtained by Kwon and Sim[1].

Corollary 2.8. Let $f \in k-ST_s^{(m)}[A, B]$. Then

$$f_m(z) = z \left(\exp(A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\epsilon^\mu z} \frac{\omega(t)}{t(1+B\omega(t))} dt \right), \quad (2.12)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Putting $\sigma = 0, m=1, A=1$ and $B=-1$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.9. Let $f \in k\text{-ST}$. Then

$$f(z) = z \cdot \left(\exp \int_0^z p_k(\omega(t)) dt - 1 \right), \quad (2.13)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Similar to the proof of Theorem 2.5, we can prove the following Theorem.

$$f(z) = \int_0^z \exp \left((A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)-1)}{t^{(B+1)p_{k,\sigma}(\omega(t))-(B-1)}} dt \right) \left(\frac{(A+1)p_{k,\sigma}(\omega(\zeta))-(A-1)}{(B+1)p_{k,\sigma}((\omega(\zeta))-(B-1))} \right) d\zeta \quad (2.15)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

$$f'(z) = \exp \left((A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)-1)}{t^{(B+1)p_{k,\sigma}(\omega(t))-(B-1)}} dt \right) \left(\frac{(A+1)p_{k,\sigma}(\omega(\zeta))-(A-1)}{(B+1)p_{k,\sigma}((\omega(\zeta))-(B-1))} \right) \quad (2.16)$$

Integrating the equation (2.16), we have

$$f(z) = \int_0^z \exp \left((A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)-1)}{t^{(B+1)p_{k,\sigma}(\omega(t))-(B-1)}} dt \right) \left(\frac{(A+1)p_{k,\sigma}(\omega(\zeta))-(A-1)}{(B+1)p_{k,\sigma}((\omega(\zeta))-(B-1))} \right) d\zeta \quad (2.17)$$

And so the proof of Theorem 2.11 is completed.

For $\sigma = 0$ we get Nasir Khan [12] result. Which reads as follows.

Corollary 2.12. Let $f \in k\text{-ST}_s^{(m)}[A, B]$. Then

$$f(z) = \int_0^z \exp \left((A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)-1)}{t^{(B+1)p_{k,\sigma}(\omega(t))-(B-1)}} dt \right) \left(\frac{(A+1)p_{k,\sigma}(\omega(\zeta))-(A-1)}{(B+1)p_{k,\sigma}((\omega(\zeta))-(B-1))} \right) d\zeta \quad (2.18)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Theorem 2.14. Let $f \in k\text{-USV}_s^{(m)}[A, B, \sigma]$. Then

$$f_m(z) = \int_0^z \left[\int_0^\zeta \exp \left((A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)-1)}{t^{(B+1)p_{k,\sigma}(\omega(t))-(B-1)}} dt \right) \right] \left(\frac{(A+1)p_{k,\sigma}(\omega(\zeta))-(A-1)}{(B+1)p_{k,\sigma}((\omega(\zeta))-(B-1))} \right) d\zeta d\zeta, \quad (2.20)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

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Theorem 2.10. Let $f \in k\text{-UCV}_s^{(m)}[A, B, \sigma]$. Then

$$f_m(z) = \int_0^z \exp \left((A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{p_{k,\sigma}(\omega(t)-1)}{t^{(B+1)p_{k,\sigma}(\omega(t))-(B-1)}} dt \right) d\zeta \quad (2.14)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

Theorem 2.11. Let $f \in k\text{-ST}_s^{(m)}[A, B, \sigma]$. Then

Proof: Let $f \in k\text{-ST}_s^{(m)}[A, B, \sigma]$. Then from equalities (2.14) and (2.15) we have

$$f(z) = \left(\frac{f_m(z)}{z} \right) \left(\frac{(A+1)p_{k,\sigma}(\omega(\zeta))-(A-1)}{(B+1)p_{k,\sigma}((\omega(\zeta))-(B-1))} \right)$$

Putting $\sigma = 0$ and $k=0$ in Theorem 2.11 we obtain the following corollary.

Which is comparable to the result obtained by Kwon and Sim[1].

Corollary 2.13. Let $f \in k\text{-ST}_s^{(m)}[A, B]$. Then

$$f(z) = \int_0^z \exp \left((A-B) \frac{1}{m} \sum_{\mu=0}^{m-1} \int_0^{\varepsilon^\mu z} \frac{\omega(t)}{t^{(1+B\omega(t))}} dt \right) \left(\frac{(1+A\omega(\zeta))}{(1+B\omega(\zeta))} \right) d\zeta \quad (2.19)$$

where $\omega(z)$ analytic function U , with $\omega(0) = 0$ and $|\omega(z)| < 1$.

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