International Journal of Scientific Engineering and Research (IJSER)
ISSN (Online): 2347-3878
Impact Factor (2022): 7.741

On Curvature Inheritance in H-⊕ Recurrent Finsler Space

S. K. Tiwari, Brijesh Kumar Maurya

Department of Mathematics, K. S. S. P. G. College, Ayodhya, Uttar Pradesh, India

sktiwarisaket[at]yahoo.com
bksir2022[at]gmail.com

Abstract: ***Singh S.P (2004), Gatoto, J.K & Singh, S.P (2008) discussed the curvature Inheritance in Finsler space under certain usual conditions and investigated necessary condition for curvature inheritance. The object of present paper is to discuss the existence of curvature inheritance in H-⊕ Recurrent Finsler space. Some significant results have been established.***

Keywords: Curvature Inheritance, Recurrent Finsler Space

1. Introduction

Consider an n-dimensional Finsler spaceF_n[4] in which the Berwald connection coefficient $G^i_{jk} = \dot{\partial}_j \dot{\partial}_k G^i$ and is based on the metric tensor $g_{ij}(x, \dot{x})$. Here $G^i(x, \dot{x})$ is positively homogeneous of degree two in its directional arguments x^i i.e

(1.1) $\dot{\partial}_h G^i \dot{x}^h = G^i_h \dot{x}^h = 2G^i$

The covariant derivative of a tensor field T^i_j with respect to x^k is given by [4]

(1.2) $T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_m T^i_j G^m_k + T^s_j G^i_{sk} - T^i_s G^s_{jk}$,

where $G^i_{mk}(x, \dot{x})$ is Berwald connection coefficient, positively homogeneous of degree zero in $\dot{x}$.

The Berwald connection G^i_{hj} satisfies the following identities-

(1.3) (a) $G^i_{hjk} \dot{x}^h = \dot{\partial}_h G^i_{jk} \dot{x}^h = 0$

(b) $G^i_{hj} \dot{x}^h = G^i_j$

(c) $G^i_{hj} = \dot{\partial}_h G^i_j$

(d) $G^i_{hjk} = \dot{\partial}_h G^i_{jk} = \dot{\partial}_j \dot{\partial}_k G^i_h$

The commutation formula arising due to covariant derivative is given by [4]

(1.4) $T^i_{j(h)(k)} - T^i_{j(k)(h)} = -\dot{\partial}_r T^i_j H^r_{hk} - T^i_s H^s_{jhk} + T^s_j H^i_{shk}$,

where,

(1.5) $H^i_{hjk} = \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^r_{hj} G^i_{ik} - G^r_{hk} G^i_{rj} + G^i_{rhk} \dot{\partial}_j G^r - G^i_{rhj} \dot{\partial}_k G^r$, is well known curvature tensor.

The tensor H^i_{jk} and H^i_k are defined as

(1.6) $H^i_{jk} = H^i_{hjk} \dot{x}^h = \partial_k \dot{\partial}_j G^i - \partial_j \dot{\partial}_k G^i + G^i_{kr} \dot{\partial}_j G^r - G^i_{rj} \dot{\partial}_k G^r$,

(1.7) $H^i_k = H^i_{jk} \dot{x}^j = 2\, \partial_k G^i - \partial_h k G^i \dot{x}^h + 2 G^i_{kl} G^l - \dot{\partial}_l G^i \dot{\partial}_k G^l$.

The curvature tensor H^i_{jhk} satisfies following identities:-

(1.8) (a) $H^i_{jhk} = -H^i_{jkh}$

(b) $H^i_{jhi} = H_{jh}$

(c) $H^i_{hjk} \dot{x}^h = H^i_{jk}$

(d) $H^i_{hjk} \dot{x}^h \dot{x}^j = H^i_k$

(e) $H^i_{jk} \dot{x}^j = -H^i_{kj} \dot{x}^j = 0$

(f) $H^i_{ji} = -H_j$.

(1.9) (a) $H^i_j \dot{x}^j = 0$

(b) $H = \frac{1}{n-1} H^i_i$

(c) $H_{jk} \dot{x}^j = H_k$.

Now we consider an n-dimensional Finsler space F_n equipped with a non-symmetric connection $\Gamma^i_{jk}(x, \dot{x}) \neq \Gamma^i_{kj}(x, \dot{x})$. Which is based on non- symmetric metric tensor $g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x})$.

Also Γ^i_{jk} is homogeneous of degree zero in its directional arguments $\dot{x}$'s.

Well known ⊕− covariant derivative of a contravariant vector $\dot{X}^i$ is given by [2]

(1.10) $X^i_{+|j} = \partial_j X^i - (\dot{\partial}_m X^i) \Gamma^m_{kj} \dot{x}^k + X^k \Gamma^i_{kj}$,

where $\Gamma^i_{jk} = M^i_{jk}(x, \dot{x}) + \frac{1}{2} N^i_{jk}(x, \dot{x})$

Here $M^i_{jk}(x,\dot{x})$ and $\frac{1}{2}N^i_{jk}(x,\dot{x})$ denote symmetric and skew-symmetric parts of Γ^i_{jk}

Also another type of covariant derivative field as $\ominus-$covariant derivative is defined as:-

(1.11) $X^i_{-|j} = \partial_j X^i - (\dot{\partial}_m X^i)\widehat{\Gamma}^m_{kj}\dot{x}^k + X^k\,\widehat{\Gamma}^i_{kj}$, [2]

where $\widehat{\Gamma}^i_{kj}(x,\dot{x}) = \Gamma^i_{kj}(x,\dot{x})$

The commutation formula involving $\oplus-$covariant derivative may be given as

(1.12) $X^i_{+|hk} - X^i_{-|kh} = -(\dot{\partial}_m X^i)R^m_{hk} + X^m R^i_{mhk} + X^i_{+|m} N^m_{kh}$, [2]

where R^h_{ijk} is corresponding curvature tensor defined by

(1.13) $R^h_{ijk} = \partial_k\Gamma^h_{ij} - \partial_j\Gamma^h_{ik} + (\dot{\partial}_m\Gamma^h_{ik})\,\Gamma^m_{sj}\dot{x}^s - (\dot{\partial}_m\Gamma^h_{ij})\,\Gamma^m_{sk}\dot{x}^s + \Gamma^p_{ij}\,\Gamma^h_{pk} - \Gamma^p_{ik}\,\Gamma^h_{pj}$,

Definition (1.1): An n-dimensional Finsler space F_n is said to be R$-\oplus$ Recurrent Finsler space if its curvature tensor R^i_{hjk} satisfies the relation [2]

(1.14) $R^i_{hjk+|s} = \beta_s R^i_{hjk}$,

where $\beta_s(x)$ denotes a non-zero covariant recurrence vector field. We shall use the following result

(1.15) $\dot{x}_{+|k} = \dot{x}_{-|k} = 0$

In analogy to definition (1.14), here we define H$-\oplus$ Recurrent Finsler space:

Definition (1.2): An n-dimensional Finsler space F_n is said to be H$-\oplus$ Recurrent Finsler space if its curvature tensor H^i_{hjk} satisfies the following relation:

(1.16) $H^i_{jkh+|m} = \beta_m(x)H^i_{jkh}$,

where $\beta_m(x)$ denotes a non-zero covariant Recurrence vector field.

Let us consider an infinitesimal point transformation

(1.17) $\dot{x}^- = x^i + v^i(x)dt$,

where $v^i(x)$ is a contravariant vector field and dt is an infinitesimal point constant

In view of infinitesimal transformation (1.17) and covariant derivative defined by (1.10) and (1.11), the Lie derivative of tensor field T^i_j and connection coefficient Γ^i_{jk} is defined as [3]

(1.18) $L_v T^i_j = T^i_{j+|k}v^k + T^i_h v^h_{-|k} - T^h_j v^i_{-|j} + \dot{\partial}_h T^i_j(v^h_{-|k}\dot{x}^k)$,

(1.19) $L_v\,\Gamma^i_{jk} = v^i_{-|j+|k} + v^h R^i_{jkh} + \dot{\partial}_r\Gamma^i_{jk}(v^r_{-|h}\dot{x}^h)$.

In view of Lie derivative (1.19), the Lie derivative of H^i_{jkh} is given by

(1.20) $L_v H^i_{jkh} = H^i_{jkh+|m}v^m + H^i_{mkh}v^m_{-|j} + H^i_{jmh}v^m_{-|k} + H^i_{jkm}v^m_{-|h} - H^m_{jkh}v^i_{-|m} + \dot{\partial}_h H^i_{jkh}(v^m_{-|r}\dot{x}^h)$.

Definition (1.3): An infinitesimal transformation (1.17) is said to define curvature inheritance if the Lies derivative of Berwald curvature tensor H^i_{jkh} satisfies the following relation:

$L_v H^i_{jkh} = \lambda H^i_{jkh}$, [5]

where $\lambda(x)$ is a non-zero scalar function.

2. H$-\oplus$ Curvature Inheritance

Definition (2.1) In H$-\oplus$ Recurrent Finsler space F^*_n, if the curvature tensor field H^i_{jkh} satisfies the relation

(2.1) $H^i_{jkh} = \alpha(x)H^i_{jkh}$,

where $\alpha(x)$ is non-zero scalar function, is called H$-\oplus$ Curvature Inheritance.

Definition (2.2): A H$-\oplus$ Recurrent Finsler space F^*_n is said to be H$-\oplus$ Curvature collineation if
(2.2) $L_v H^i_{jkh} = 0$.

Definition (2.3): A Finsler space in which Berwald curvature tensor H^i_{jkh} satisfies the relation –

$H^i_{jkh+|m} = o$,

is called H$-\oplus$ symmetric Finsler space.

Consider an infinitesimal concircular transformation
(2.4) $\bar{x}^i = x^i + v^i dt,\ v^i_{-|j} = \lambda\delta^i_j$
where λ is non-zero constant.

In view of existence of curvature Inheritance and result (1.16), (1.20) becomes

$\alpha(x)H^i_{jkh} = \beta_m v^m H^i_{jkh} + H^i_{mkh}v^m_{-|j} + H^i_{jmh}v^m_{-|k} + H^i_{jkm}v^m_{-|h} - H^m_{jkh}v^l_{-|m} + \dot{\partial}_m H^i_{jkh}(v^m_{-|r}\dot{x}^r)$
$= \beta_m v^m H^i_{jkh} + H^i_{mkh}\lambda\delta^m_j + H^i_{jmh}\lambda\delta^m_k + H^i_{jkm}\lambda\delta^m_h - H^m_{jkh}\lambda\delta^l_m + \dot{\partial}_m H^i_{jkh}(\dot{x}^r\lambda\delta^m_r)$

due to homogeneity condition of H^i_{jkh}, above equation reduces to

$\alpha(x)H^i_{jkh} = \beta_m v^m H^i_{jkh} + H^i_{jkh}\lambda + H^i_{jkh}\lambda + H^i_{jkh}\lambda - H^i_{jkh}\lambda + \dot{\partial}_m H^i_{jkh}(\dot{x}^r\lambda\delta^m_r)$
$=\beta_m v^m H^i_{jkh} + 2H^i_{jkh}\lambda$

(2.5) $\left[\alpha(x) - \beta_m v^m - 2\lambda\right] H^i_{jkh} = 0$

For non-flat Finsler space, $H^i_{jkh} \neq 0$ so (2.5) becomes

(2.6) $\alpha(x) = \beta_m v^m + 2\lambda$

Theorem (2.1): The necessary condition for existence of concircular curvature inheritance in $H-\oplus$ Recuurnt Finsler space is given by (2.6)

If we take $H-\oplus$ symmetric Finsler space then transvecting (2.3) by v^m, we get

(2.7) $v^m H^i_{jkh+|m} = 0$

If $\beta_m = 0$ i.e $H-\oplus$ recurrent Finsler space becomes $H-\oplus$ symmetric Finsler space, then

(2.8) $\alpha = 2\lambda$.

Corollary: In $H-\oplus$ Symmetric Finsler space, the concircular transformation admits curvature Inheritance and non-zero scalar function

$\alpha = 2\lambda$.

3. Curvature Inheritance in special Recurrent Finsler space

Consider the vector field v^i in the following Recurrent form

(3.1) $v^i_{-|j} = \psi_j v^i$,

where $\psi_j(x)$ denotes an arbitrary covariant vector and the field spanned by above form is called Recurrent vector field.

Now we consider an infinitesimal transformation

(3.2) $\bar{x}^i = x^i + v^i dt, v^i_{-|j} = \psi_j v^i$

Such transformation is called Recurrent transformation.

In view of Recurrent transformation(3.2) and (1.14), (1.12) reduces to

(3.3) $$L_v H^i_{jkh} = \beta_m v^m H^i_{jkh} + H^i_{mkh}\left(\psi_j v^m\right) + H^i_{jmh}\left(\psi_k v^m\right) + H^i_{jkm}(\psi_h v^i) - H^m_{jkh}(\psi_m v^i) + \dot{\partial}_m H^i_{jkh}(\psi_r v^m \dot{x}^r)$$
$$= \beta_m v^m H^i_{jkh} + (H^i_{mkh}\psi_j + H^i_{jmh}\psi_k + H^i_{jkm}\psi_h)v^m - H^m_{jkh}(\psi_m v^i) + \dot{\partial}_r(H^i_{jkh}\psi_r)v^m \dot{x}^r$$
$$= \beta_m v^m H^i_{jkh} + (H^i_{mkh+|_j} + H^i_{jmh+|_k} + H^i_{jkm+|_h} + \dot{\partial}_r H^i_{jkh+|_r}\dot{x}^r)v^m - H^m_{jkh}(\psi_m v^i)$$

(3.4) $\Rightarrow L_v H^i_{jkh} \neq 0$.

Thus we have,

Theorem (3.1) In $H-\oplus$ recurrent Finsler space, recurrent transformation (3.2) admits curvature Inheritance.

If $H-\oplus$ recurrent Finsler space admits curvature inheritance then (3.3) becomes

(3.5) $$(\alpha - \beta_m v^m)H^i_{jkh} = \beta_m v^m H^i_{jkh} + (H^i_{mkh+|_j} + H^i_{jmh+|_k} + H^i_{jkm+|_h} + \dot{\partial}_r H^i_{jkh+|_r}\dot{x}^r)v^m + H^m_{jkh}\psi_j v^i.$$

Theorem (3.2) If $H-\oplus$ recurrent Finsler space admits curvature inheritance subjectto recurrent transformation given by (3.2) then (3.5) holds good.

References

[1] Gatoto, J.K. and Singh, S.P.(2008) On K-curvature inheritance in Finsler space, Tensor N.S., 70 1-7
[2] Pande, H.D. and Gupta, K.K., An $R-\oplus$ recurrent Finsler space with non-symmetric connection, Instanbul Univ., Fen. Seri., A, 43 (1978), 107-112
[3] Pande, H.D. and Tiwari, S.K.: Generalised Affine Motion in a $R-\oplus$ recurrent Finsler space
[4] Rund, H.: The Differential geometry of Finsler spaces, Springer-Verlag, Berlin (1939)