

Generalization of Colinear Collision Model Development Using the Admittance Analogy

Jai Singh

Biomechanical Engineering Analysis & Research, Inc.
2060-D Avenida de los Arboles, No. 487, Thousand Oaks, California 91362, USA
[jaisingh\[at\]bearinc.org](mailto:jaisingh[at]bearinc.org)

Abstract: Collisions that are inherently colinear in nature or that can be modeled as such represent a common configuration encountered in accident reconstruction practice. When mathematical models, based upon the relationship between the time parametric collision force and deflection response are utilized, they tend towards simplicity, based, in part, upon the typical scope and extent of evidence that is readily available following such collisions under field and subsequent conditions. Within the boundaries of these limitations, one may readily posit models of greater complexity, which in turn allow for the modeling of additional collision associated phenomenon. Presented in the subject work, predicated upon the admittance and force-current analogies, is a generalized solution for the determination of the Laplace domain transfer function, with simultaneous determination of the Laplace domain forcing functions secondary to initial conditions. Furthermore, the generalized solution is appropriate for both the closure and the separation phases of a collision. This generalized solution is appropriate for uniaxial collision modeling for a system consisting of two discrete masses connected in series at a massless interface by means of linear springs and linear dampers. The general solution is validated by comparing the results generated for four distinct cases against the results, for each case, based upon evaluation of the underlying differential equations of motion and force balance constraint equation.

Keywords: Accident reconstruction, colinear collision analysis, impedance modeling, Laplace domain

1. Introduction

Efficacious mathematical modeling of physical phenomenon serves as a method by which the complexity of the latter is reduced to tractability while retaining saliency in regards to the relevant system input parameters, output parameters and interrelationships. One common approach, utilized across multiple engineering domains, is that of lumped parameter modeling. Conceptually, this modeling approach involves (a) determining a finite set of domain specific and relevant system variables, (b) reducing the degrees of freedom to a finite set, (c) assigning the domain specific and generally distributed set of model parameters to a discrete set of model elements and (d) relating the model parameters to the system variables.

As a very simple example, one may consider the case of an idealized simple harmonic oscillator. This example is chosen for two reasons. The first is that one has a reasonable expectation that most readers will have an *a priori* familiarity with the model given its inclusion in introductory physics courses (covering mechanics), introductory single variate calculus courses and introductory differential equations courses. The second reason is that the example allows for the introduction of translational mechanics domain specific and application specific terminology.

The idealized simple harmonic oscillator is a constrained form of a slightly more complex system formulation. We first note that the problem is uniaxial, in translation, and collinear, as applicable. This is followed by noting that all coordinate axes are aligned at the temporal start of consideration (i.e. at $t = t_0 = 0$). These two considerations reduce the generally time-varying direction cosine matrices that relate vector components between different frames of references to time-invariant identity matrices. This, in turn,

means that all system translational mechanics variables are referenced with respect to the inertial frame of reference.

Shown in Figure 1 is an idealized model consisting of two discrete masses (m_1 and m_2) that are connected to each other by means of massless elements (denoted as k_1 and k_2) at a massless node. Vectors are shown in boldface font and simple time derivatives are denoted using the standard overdot notation. The time-varying displacements of the two masses are respectively denoted as $\mathbf{u}_a(t)$ and $\mathbf{u}_c(t)$. The time-varying displacement of the massless interface is denoted as $\mathbf{u}_b(t)$.

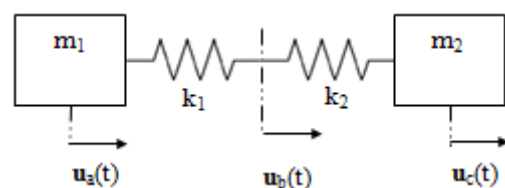


Figure 1: Idealized two degree of freedom system consisting of two lumped masses (m_1 and m_2) connected by means of two massless elements (denoted as k_1 and k_2) at a massless interface.

The force generated within each massless element is modeled as being a linear function of the relative displacement between the nodes that represent the endpoints of the element. When this relative displacement results in a change in length of the element, the term deflection is apt. Finally, we note that each such element has a reference length at which it carries zero force. For this example the length of each element at time $t = t_0$ is the reference length. The time-varying force developed in each element is thusly expressed as the following:

$$\begin{aligned} \mathbf{F}_1(t) &= k_1(\mathbf{u}_a(t) - \mathbf{u}_b(t)) = k_1\mathbf{u}_{ab}(t) = k_1\boldsymbol{\delta}_1(t) \\ \mathbf{F}_2(t) &= k_2(\mathbf{u}_b(t) - \mathbf{u}_c(t)) = k_2\mathbf{u}_{bc}(t) = k_2\boldsymbol{\delta}_2(t) \end{aligned} \quad (1)$$

$$\mathbf{u}_1(s) = \frac{\dot{\mathbf{u}}_{1o}}{s^2 + \omega_1^2} \quad (7)$$

This slightly more complex example is covered in greater detail in a subsequent section. For now, it is sufficient to reduce it to the case of the idealized simple harmonic oscillator. This is done by noting that the object represented by m_2 and k_2 is an ideal fixed, rigid, massive barrier (FRMB). In this idealization the relationship of $k_2 \gg k_1$ can be viewed as $k_2 \rightarrow \text{infinity}$. This rigidity means that $\mathbf{u}_{bc}(t) = 0$. Because the ideal wall is fixed and massive, $\mathbf{u}_c(t) = 0$, which as per the previous statement means that $\mathbf{u}_b(t) = 0$. For this case, the common massless interface is simply the face of the wall. Finally, based on (1), $\boldsymbol{\delta}_1(t) = \mathbf{u}_{ab}(t) = \mathbf{u}_a(t) - \mathbf{0} = \mathbf{u}_a(t)$. Thusly, for this case, the deflection experienced by the element k_1 is equal to the displacement experienced by the mass m_1 . This finding does not hold for the general case in which the object represented by k_2 and m_2 is deformable, displaceable or both. The force experienced by m_1 can readily be determined by applying Newton's second and third laws.

$$m_1\ddot{\mathbf{u}}_1(t) = -\mathbf{F}_1(t) = -k_1\mathbf{u}_1(t) \quad (2)$$

Rewriting (2) by algebraic rearrangement and by noting that ω_1 , the circular frequency, is equal to the square root of k_1/m_1 , leads to the operative second order linear differential equation of motion for the idealized simple harmonic oscillator.

$$\ddot{\mathbf{u}}_1(t) + \omega_1^2\mathbf{u}_1(t) = \mathbf{0} \quad (3)$$

This equation is subject to the initial conditions of zero initial acceleration, finite non-zero initial velocity and zero initial displacement.

$$\ddot{\mathbf{u}}_1(t_o) = \dot{\mathbf{u}}_{1o} = \mathbf{0} \quad \dot{\mathbf{u}}_1(t_o) = \dot{\mathbf{u}}_{1o} \quad \mathbf{u}_1(t_o) = \mathbf{u}_{1o} = \mathbf{0} \quad (4)$$

There are a number of ways to solve (3), subject to the initial conditions shown by (4). The preferred method in regards to the subject work, and a method that is employed ubiquitously in this work, is the use of the unilateral (i.e. one-sided) Laplace transform. This integral transform, for a function $f(t)$, for $t \geq 0$, results in a function $F(s)$, which is defined by (5).

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (5)$$

A utility of this transform is that it converts linear differential equations into algebraic equations. The Laplace transform for the first and second time derivatives of a function $f(t)$, which is assumed to be differentiable and of the exponential type, are given in (6).

$$\begin{aligned} L(f'(t)) &= sF(s) - f_o \\ L(f''(t)) &= s^2F(s) - sf_o - f'_o \end{aligned} \quad (6)$$

One may readily find tables that detail the Laplace transform of various functions along with the corresponding inverse Laplace transforms. Applying (6) to (3) while making use of (4) leads to the following result after algebraic rearrangement.

The inverse Laplace transform of (7) yields the time domain solution for the displacement $\mathbf{u}_1(t)$. This solution can be differentiated, with respect to time, to yield the time domain velocity solution and differentiated with respect to time, again, to yield the time domain acceleration solution.

$$\begin{aligned} \mathbf{u}_1(t) &= \frac{\dot{\mathbf{u}}_{1o}}{\omega_1} \sin(\omega_1 t) \\ \dot{\mathbf{u}}_1(t) &= \dot{\mathbf{u}}_{1o} \cos(\omega_1 t) \\ \ddot{\mathbf{u}}_1(t) &= -\omega_1 \dot{\mathbf{u}}_{1o} \sin(\omega_1 t) \end{aligned} \quad (8)$$

The kinematic response for the idealized simple harmonic oscillator is obviously oscillatory. The frequency of oscillation is given by $f = \omega_1/(2\pi)$ and the period of oscillation is given by $T = 1/f = (2\pi)/\omega_1$. This oscillatory nature can also be appreciated by examining the operands of the sine and cosine functions. Both the displacement and acceleration are zero valued when the operand of the sine function is $n\pi$, where n is any integer value. The peak positive displacement and peak negative acceleration magnitudes occur when the operand of the sine function reduces to 0.5π . The converse holds when the operand of the sine function reduces to 1.5π . The velocity is 0.5π radians out of phase with respect to the displacement. The peak positive value of the velocity occurs when the operand of the cosine function reduces to zero. The peak negative value of the velocity occurs when the operand of the cosine function reduces to $n\pi$, wherein n is an odd integer. The velocity solution is zero valued at $0.5n\pi$, where n is an odd integer. The absolute values of the peaks, for each kinematic response, are equal.

One may consider the question of the utility of such a model when it comes to modeling collisions. In order to do so, one must first introduce some additional terminology. We define a collision as a physical event in which two or more discrete objects or discrete regions of the same object attempt to contemporaneously occupy the same region of physical space. Collisions occur over a finite, albeit generally short, temporal duration. For certain collisions, however, one may employ mathematical models that are based upon a single time step approach. For the purpose of the subject work, the collisions of interest are first limited to those for which the mass of the collision partners is time-invariant, those in which the materials constituent within the colliding objects experience no phase change and collisions in which substantial thermal effects are negligible. Furthermore, the collisions of interest are further limited to those that are ubiquitously uniaxial and collinear.

Excluding sideswipe type collisions, which may partially fit this characterization, the collisions of interest are temporally divisible into an initial closure phase followed by a subsequent separation phase. The former initiates at the first moment the collision partners come into contact and terminates at the moment when the collision partners achieve a common velocity (at time $t = t_c$). The latter initiates contemporaneously with the terminus of the closure phase

and terminates at the first moment that the collision force returns to zero magnitude (at time $t = t_c$). During the closure phase, internal work is absorbed (IWA) by the collision partners. During the separation phase, typically only a portion of the IWA is recovered (IWR) and with the remainder being dissipated (IWD).

Returning to the question that was asked in the previous paragraph, it can clearly be stated that the idealized simple harmonic oscillator, *en toto*, is inappropriate as a collision model simply due to its *ad infinitum* oscillatory nature. If the full, unmodified model is inappropriate, would a temporally limited model be appropriate? It can readily be stated that a full period model would be inappropriate secondary to the occurrence of one full oscillation and with a return of the initial conditions. A half period model is appropriate for a very idealized collision – one for which the IWR equals the IWA and thusly provides for a unity (absolute) valued coefficient of restitution (ratio of the separation velocity to the closing velocity). A further temporal restriction, that being the first quarter period, for modeling the closure phase alone, for certain collisions, greatly expands the scope of potential applicability for the model. The reference to certain collisions clearly refers to the case in which a collinear impact occurs between the end of an object, such as a motor vehicle, and a FRMB. Such an impact aptly describes the collision configuration utilized in the United States (US) Federal Motor Vehicle Safety Standard (FMVSS) 208 dynamic (208D) front impact compliance testing protocol, high speed front impact New Car Assessment Program (NCAP) testing protocol and other similar testing conducted for research purposes, test development purposes or both. Furthermore, such a model is apt when (a) a single degree of freedom (SDOF) is sufficient and (b) when the peak collision force and peak deflection occur within the vicinity of the terminus of closure phase of the collision. The reasoning for the first caveat is patent given that the model is a SDOF model. The reasoning for the second can be seen from the following. The modeled time at which the closure phase terminates is $t_c = \pi/(2\omega_1)$. This solution derives from the fact that the common velocity is zero given the nature of the FRMB. The operand of the cosine function first reaches a zero valuation at the above-noted value. The ratio of the collision force to the deflection experienced by the front of the test vehicle during the closure phase can readily be seen from (1) or calculated from (2) and (8). From (1), it can readily be seen that the collision force is a linear, monotonically increasing, function of deflection. It should also be clear that the model, for the closure phase, is a single parameter model. When working with data from any given collision test, the time at which the velocity of the test vehicle reaches a zero valuation is readily determinable. This allows for the direct determination of the time at which closure terminates, which in turn allows for one method of determining ω_1 by $\omega_1 = \pi/(2t_c)$. Since the initial velocity of the test vehicle is measured, one can determine modeled values for the peak deflection and peak acceleration magnitude at the terminus of closure. One may also evaluate the problem in the following manner. The IWA, irrespective of the force-deflection model used, for a collinear collision, can be shown to be the following (where v_c is the closing

velocity, which is the difference in the velocities of the collision partners at the start of the closure phase) [1].

$$IWA = \frac{1}{2} m_{red} v_c^2 = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v_c^2 \xrightarrow[\frac{m_2 \rightarrow \infty}{\dot{u}_{30} = 0}]{\rightarrow} \frac{1}{2} m_1 \dot{u}_{10}^2 \quad (9)$$

The IWA is also equal to the integral of the dot product of the collision force and the differential of its work conjugate displacement (here, $d\mathbf{u} = d\delta_1 = d\mathbf{u}_1 - d\mathbf{u}_2 = d\mathbf{u}_1$). For the linear closure phase model, this can readily be calculated as the following (where \mathbf{F}_c is the collision force at the terminus of the closure phase):

$$IWA = \frac{1}{2} k \delta_{1c}^2 = \frac{1}{2} \mathbf{F}_c \delta_{1c} \quad (10)$$

One may use the known initial velocity and determined value for δ_{1c} to solve for ω_1 from the displacement solution under (8) and to solve for a modeled value for \mathbf{F}_c by equating (9) and (10) and solving. This linear model for the closure phase has been used almost ubiquitously in the accident reconstruction engineering literature [2-4].

The next question that logically follows is can this model, under the aforementioned caveats, be used for modeling the separation phase? Clearly, one would want a model with differing parameter (i.e. stiffness and circular frequency) valuations due to the fact that a model with the same closure phase parameter valuations would simply provide the second quarter of a half period idealized simple harmonic oscillator response with full energy recovery and a coefficient of restitution magnitude of unity. Thusly the separation phase values are denoted using the overbar notation. Secondly, one would rather not have $t - t_c$ within the parenthetical time dependance. Thusly, for the sake of clarity, a new temporal variable, $\tau = t - t_c$, is introduced for the separation phase. As a result $\tau_o = t_c - t_c$ and $\tau_s = t_s - t_c$. Finally, one must consider the form of the second order differential equation of motion. The solution for all three kinematic responses must be continuous at $t = t_c$ from both the closure phase and separation phase approaches. This is achieved by rewriting (3) as follows:

$$\ddot{\mathbf{u}}_1(\tau) + \bar{\omega}_1^2 \mathbf{u}_1(\tau) = \ddot{\mathbf{u}}_{1c} + \bar{\omega}_1^2 \mathbf{u}_{1c} \quad (11)$$

The initial conditions for this equation are the terminus of closure phase values.

$$\ddot{\mathbf{u}}_1(\tau_o) = \ddot{\mathbf{u}}_{1c} \quad \dot{\mathbf{u}}_1(\tau_o) = \mathbf{0} \quad \mathbf{u}_1(\tau_o) = \mathbf{u}_{1c} \quad (12)$$

Taking the unilateral Laplace transform of (11), subject to the initial conditions given by (12) and algebraically rearranging the results leads to the following Laplace domain solution for the displacement.

$$\mathbf{u}_1(s) = \frac{1}{s^2 + \bar{\omega}_1^2} \left(\frac{\ddot{\mathbf{u}}_{1c} + \bar{\omega}_1^2 \mathbf{u}_{1c}}{s} + s \mathbf{u}_{1c} \right) \quad (13)$$

The time-domain kinematic solutions for the separation phase, for this model, can readily be obtained as before (for the closing phase model).

$$\begin{aligned} \mathbf{u}_1(\tau) &= \mathbf{u}_{1c} + \frac{\ddot{\mathbf{u}}_{1c}}{\omega_1^2} (1 - \cos(\bar{\omega}_1 \tau)) \\ \dot{\mathbf{u}}_1(\tau) &= \frac{\dot{\mathbf{u}}_{1c}}{\omega_1} \sin(\bar{\omega}_1 \tau) \\ \ddot{\mathbf{u}}_1(\tau) &= \ddot{\mathbf{u}}_{1c} \cos(\bar{\omega}_1 \tau) \end{aligned} \quad (14)$$

It can readily be seen that each response correctly predicts the terminus of closure phase values at $\tau = 0$. For the separation phase, the solution for the time at which separation terminates is determined by setting the acceleration in (14) equal to zero and solving for τ .

$$\tau_s = \frac{\pi}{2\bar{\omega}_1} \rightarrow t_s = t_c + \frac{\pi}{2\bar{\omega}_1} \quad (15)$$

Substitution of the solution from (15) into the displacement and velocity solutions from (14) result in the following modeled terminus of separation phase results.

$$\mathbf{u}_{1s} = \mathbf{u}_{1c} + \frac{\ddot{\mathbf{u}}_{1c}}{\omega_1^2} \quad \dot{\mathbf{u}}_{1s} = \frac{\dot{\mathbf{u}}_{1c}}{\omega_1} \quad (16)$$

For the test type under question and the sign convention employed, the terminus of closure acceleration will be negatively signed, thereby resulting in $\mathbf{u}_{1s} < \mathbf{u}_{1c}$ and negatively signing the velocity of the test vehicle at terminus of separation. As with the linear closure phase model, the linear separation phase model is a single parameter model. One may readily solve for τ_s from actual data and then solve for the circular frequency using (15). One may also use work-energy relationships. The IWR, irrespective of the force-deflection model used, for a collinear collision, can be shown to be the following (where \mathbf{v}_s is the separation velocity, which is the difference in the velocities of the collision partners at the terminus of the separation phase) [1].

$$IWR = \frac{1}{2} m_{red} \mathbf{v}_s^2 = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \mathbf{v}_s^2 \xrightarrow[\substack{\dot{\mathbf{u}}_{2s}=0 \\ m_2 \rightarrow \infty}]{\dot{\mathbf{u}}_{1s}=\dot{\mathbf{u}}_{1c}} \frac{1}{2} m_1 \dot{\mathbf{u}}_{1c}^2 \quad (17)$$

The IWR may also be determined from the linear force-deflection model in the same way as it was for the closure phase.

$$IWR = \frac{1}{2} \bar{k}_1 (\delta_{1c} - \delta_{1s})^2 = \frac{1}{2} \mathbf{F}_c (\delta_{1c} - \delta_{1s}) \quad (18)$$

One may readily solve for the model parameters and values using the same methods as discussed for the closure phase. As an aside, one finds that the linear deflection only closure phase model has greater applicability than the linear deflection only separation phase model. The straight line fit between the points $(\delta_{1c}, \mathbf{F}_c)$ and $(\delta_{1s}, \mathbf{0})$ generally tends to over-predict the IWR. This is due to the fact that the separation phase response, even for the SDOF model, for data from high speed collision testing, tends to be highly nonlinear (with respect to a straight line fit for the force-deflection response) with an initial phase of finite but high stiffness followed by a second phase of substantially lower stiffness. Methods regarding model parameter estimation and the relationship between dynamic modeling and quasi-empirical residual damage based modeling have been addressed elsewhere [4-5] and are not the focus of the subject work.

The focus of the subject work can, first, be seen by example

in the comparison of (7) and (13). Both equations are in the form of an output (the Laplace domain displacement) being equal to a Laplace domain function multiplied by an input (the Laplace domain forcing function based upon the initial conditions of the system). Both equations can be written in a general form based upon the ratio of Laplace domain output to the Laplace domain input.

$$H(s) = \frac{\mathbf{u}_1(s)}{\mathbf{F}(s)} \quad (19)$$

In (19), $H(s)$ is known as the transfer function. There are two observations of import to the subject work. The first is that the form of $H(s)$ in (7) and (13) is the same for both the closure phase and the separation phase (i.e. the inverse of s^2 plus the square of the operative circular frequency). The second is that this form remains unchanged irrespective of the form of the forcing function. One may readily consider the case, for example, in which a net unbalanced externally applied load is applied to m_1 . If the function has a Laplace transform then the effect on the displacement response can simply be obtained by adding the multiple of the same and the transfer function to the Laplace domain displacement solution. The objective of the subject work, thusly, can be stated as the theoretical development of a methodology for generalizing the development of the form of the transfer function for certain common model configurations for the two collision partner collinear collision case in which each collision partner is modeled using basic linear lumped element components.

2. Theory

2.1 System Generalization

Systems of the subject type can readily be generalized based upon variable pairs whose product is equal to power [6]. This generalization is applicable to a number of contexts that include but are not limited to translational mechanical systems, rotational mechanical systems, electrical systems and thermal systems. System variables are classified as either being across variables or through variables. Across variables are those that are defined by measuring their change across an element while through variables are those that are transmitted, unchanged, through an element. Across variables are subject to a compatibility condition, which states that the sum of the variables across any number of elements in a closed loop is equal to zero. Through variables are subject to a continuity condition, which states that the sum of the variable through any number of elements connected at a node (i.e. a connection within the system topology) is zero.

One may also define two generalized sources [7]. The first source, is a generalized across variable source, which maintains the value of the across variable irrespective of the magnitude of the applied through variable. The second source, a generalized through variable source, maintains the prescribed magnitude of the through variable to its connected nodes irrespective of the magnitude of the across variable that must be generated to maintain the prescribed magnitude.

Finally, one may define three types of primitive elements that consist of two energy storage elements and one dissipative element. A-type elements are those that store energy as a function of the across-variable. Letting ϕ denote a generalized through variable, η denote a generalized across variable and letting χ be a constant, the following relationship holds for an A-type element.

$$\phi = \chi \frac{d\eta}{dt} \quad (20)$$

The second element type, the T-type element, is an energy storage element that is a function of a through variable. Using the definitions from above leads to the following relationship.

$$\eta = \chi \frac{d\phi}{dt} \quad (21)$$

The third element type, the D-type element, is a dissipative element (power ≥ 0). The generalized relationship for this element is given as follows.

$$\eta = \chi\phi \quad (22)$$

We may also define the generalized impedance, $Z(s)$, and its inverse, the generalized admittance, $Y(s)$, in the following manner.

$$Z(s) = \frac{\eta(s)}{\phi(s)} \quad Y(s) = \frac{\phi(s)}{\eta(s)} \quad (23)$$

2.2 Force-current analogy

Moving from generalization to specificity, it is first important to consider electrical circuit systems secondary to the same serving as an underlying basis for presenting analogous mechanical systems [8]. For electrical systems, the power conjugate variables (i.e. the variables whose product equals power) are voltage drop (V) and current (i). If $V = V_a - V_b$ is the voltage drop across a capacitor with capacitance C (an A-type element), then the current through the capacitor is given by the following.

$$i = C \frac{dV}{dt} \quad (24)$$

The energy stored by the capacitor is given by the following.

$$E = \int_{-\infty}^t V i dt = \int_0^t C V dV = \frac{1}{2} C V^2 \quad (25)$$

If L is the inductance of an inductor and V is the voltage drop across the inductor (a T-type element), then the relationship between the voltage drop and current is given by the following.

$$V = L \frac{di}{dt} \quad (26)$$

The energy stored by the inductor is given by the following.

$$E = \int_{-\infty}^t V i dt = \int_0^t L i di = \frac{1}{2} L i^2 \quad (27)$$

Finally, if R is the resistance of a resistor and V is the voltage drop across the resistor (a D-type element), then the relationship between the voltage drop and the current is given by the following.

$$V = iR \quad (28)$$

The power dissipated by the resistor is given by the following.

$$P = V i = i^2 R \geq 0 \quad (29)$$

It should be readily apparent that (24) and (26) are invertible by multiplying both sides of each equation by the differential, dt , and integrating. For a translational mechanical system, the system power conjugate variables are force and velocity. One may analogize the translational mechanical system to an electrical system by analogizing velocity with current (and thereby force with voltage) [9]. Alternatively, one may analogize force with current (and thereby velocity with voltage) [10]. Both analogies are correct but lead to differing formulations. Great care should be taken when comparing sources within the literature to ensure that one understands the analogy being used. For the subject work, the force-current analogy is utilized. As a result, mass is analogous to capacitance, linear springs are analogous to inductors and linear dampers are analogous to resistors. Because of this choice of analogy, the sum of velocity drops around a closed loop (see the next section), is the compatibility condition analogous to Kirchoff's voltage law (KVL). The continuity condition analogous to Kirchoff's current law (KCL), is that the sum of forces at any node (see the next section), including the d'Alembert force for a mass element, is zero. The Laplace domain impedances for a capacitor, inductor and resistor of $1/(Cs)$, sL and R , respectively, for the force-current analogy, become the following impedances for mass, linear spring and linear damper elements (where $Z(s) = v(s)/F(s)$ for translational mechanical system).

$$Z_m = \frac{1}{sm} \quad Z_k = \frac{s}{k} \quad Z_c = \frac{1}{c} \quad (30)$$

The corresponding admittance values for the same are given by the following.

$$Y_m = sm \quad Y_k = \frac{k}{s} \quad Y_c = c \quad (31)$$

For this analogy, (23) can be expressed in the following manner.

$$Z(s) = \frac{\mathbf{v}(s)}{\mathbf{F}(s)} \quad Y(s) = \frac{\mathbf{F}(s)}{\mathbf{v}(s)} \quad (32)$$

The choice of analogy also plays a role in regards to the combination of elements in series and parallel. For the force-current analogy, elements in series share the same force and the equivalent impedance is the sum of the impedances of the elements in series.

$$Z_{eq}(s) = \sum_{i=1}^n Z_i(s) \quad (33)$$

Elements in parallel, for this analogy, share the same velocity and as a result the equivalent impedance is the inverse of the sum of inverse of the impedances of the elements in parallel.

$$\frac{1}{Z_{eq}(s)} = \sum_{i=1}^n \frac{1}{Z_i(s)} \quad (34)$$

The relationships are inverted when one replaces the impedance terms with the corresponding admittance terms. Finally, we note that for the force-current analogy, forcing functions are analogous to current sources and velocity sources are analogous to voltage sources.

2.3 Line graphs

Line graphs are a simple yet effective tool for visualizing the structural architecture and connectivity of dynamic systems [11]. Care, again, must be taken in comparing presentations of this topic across references secondary to the fact that differing authors will use differing conventions. With this fact noted, we define nodes in a line graph as points of connectivity, with specifiable velocities (i.e. nodal velocities). The connections between the nodes are referred to as branches.

System impedances are shown as being located along the branches. An arrow showing both the presumed direction of velocity drop between the connected nodes and force flow, is used in conjunction with an indication of the element. Two points need to be noted regarding mass elements. The first is that mass elements always have a branch connected to the node serving as the inertial frame of reference. This is because of the specification that all forces and kinematic parameters are properly referenced to the inertial frame of reference. The velocity of this inertial frame of reference is specified as zero. The second is that the direction of force flow along the branch connecting a mass element towards the reference node is always directed towards the reference node (due to the sign of the d'Alembert force in the continuity equation).

Force sources, analogous to current sources, are shown as circles with a directional arrow for the flow of force, along branches. Velocity sources are not used in the subject work.

The analogies to KVL and KCL were noted previously. In the subject work velocity drops are taken in a clockwise manner and sum to zero for any closed loop. For example, for the closed loop defined by the clockwise arranged nodes a, b and c, the velocity drops are $v_{a|b} + v_{b|c} + v_{c|a} = (v_a - v_b) + (v_b - v_c) + (v_c - v_a) = 0$. For the analog to KCL, the sum of forces flowing into a node is equal to the sum of forces flowing out of a node.

As an example, we first consider a slightly more complicated version of the SDOF case that was detailed previously. Consider the case of a single lumped mass connected to a displacement constraint by means of a linear relative displacement and a linear relative displacement rate element. Let k denote the stiffness of the former and let c denote the damping coefficient of the latter. We also consider a time-varying force $F(t)$, for which the Laplace transform, $F(s)$, exists, acting on the mass. This is shown in Figure 2 along with the corresponding line graph.

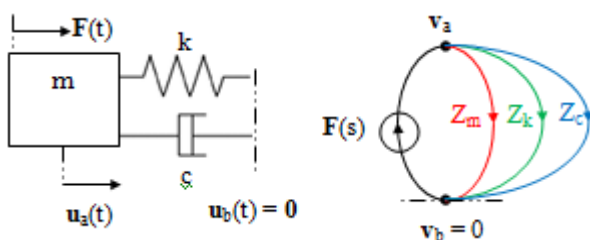


Figure 2: Shown on the left is an example of a SDOF damped system consisting of a lumped mass (m), stiffness

(k) and damper (c) with force $F(t)$ acting upon the mass. Shown on the right is the corresponding line graph.

The system configuration shown in Figure 2 clearly would not be appropriate for a closure phase model due to the fact that the presence of the damper would result in the greatest force from the same being present at the start of closure (due to the velocity difference with the displacement constraint being the greatest at that time) and thusly the model would predict a non-zero valued force at $t = t_0$. Setting this issue aside, the line graph, rather than the standard schematic, more clearly shows that the three passive elements of the system are in parallel. The equivalent impedance of these three elements can be obtained from using the relationships given in (30) and (34).

$$Z_{eq}(s) = \left(\frac{1}{Z_m} + \frac{1}{Z_k} + \frac{1}{Z_c} \right)^{-1} = \left(ms + \frac{k}{s} + c \right)^{-1} \tag{35}$$

$$= \frac{s}{ms^2 + cs + k}$$

The velocity drop across the equivalent impedance is $v_{a|b} = v_a - 0 = v_a$. From (32), $Z_{eq}(s) = v_{a|b}(s)/F_{eq}(s) = v_a(s)/F_{eq}(s)$. The sum of forces at node (a) yields $F(s) = F_{eq}(s) = v_a(s)/Z_{eq}(s)$. The relationship between displacement and velocity in the Laplace domain is $u(s) = v(s)/s$. As a result, the transfer function for this example is the following.

$$H(s) = \frac{u_a(s)}{F(s)} = \frac{1}{s} \frac{v_a(s)}{F(s)} = \frac{1}{s} Z_{eq}(s) = \frac{1}{ms^2 + cs + k} \tag{36}$$

The result shown in (36) is the expected transfer function for a second order damped equation of motion. One could have readily obtained the same result by taking the Laplace transform of the time domain second order differential equation of motion under the condition of all initial conditions being zero valued. Ideally, however, for the subject work, we would like to generate both the transfer function and the general form of the forcing function due to the initial conditions in a single derivation rather than having to determine the latter at a later point. An approach for doing so is shown in the following section.

2.4 Initial conditions

The example shown in Figure 2 is a good example for introducing the manner for considering system initial conditions. The following presentation is equally applicable to the idealized simple harmonic oscillator discussed previously (by setting c equal to zero). To generalize the presentation we expand the definition of the temporal variable, τ , such that $\tau = t - t_a$. The time value of $t = t_a$ can be at any point during the event.

Based upon the prior definition, $\tau_a = t_a - t_a = 0$. Furthermore, we define τ_b such that $\tau_a = 0 \leq \tau \leq \tau_b$ with the limitation that c and k do not change over the temporal duration. For a collision, the temporal limitation means that τ_a and τ_b operate over any portion of the closure phase (fully inclusive), any portion of the separation phase (fully inclusive) or any other temporal region for any other formulation over which the model parameters do not change value. This point is

mentioned not because the model form, as presented below, changes, but rather to remind the reader that the values used for the model parameters are those operative over a given phase and the initial conditions are those that exist at τ_a . With these caveats noted, we start by writing the second order differential equation of motion for the system shown in Figure 2. This form is similar to (11) but includes a term due to the damper (also, the displacement \mathbf{u}_a is change to \mathbf{u} and the subscript of 'a' is used to denote the value of the corresponding variable at τ_a).

$$m\ddot{\mathbf{u}}(\tau) + c\dot{\mathbf{u}}(\tau) + k\mathbf{u}(\tau) = m\ddot{\mathbf{u}}_a + c\dot{\mathbf{u}}_a + k\mathbf{u}_a \quad (37)$$

The system initial conditions are as follows:

$$\ddot{\mathbf{u}}(\tau_a) = \ddot{\mathbf{u}}_a \quad \dot{\mathbf{u}}(\tau_a) = \dot{\mathbf{u}}_a \quad \mathbf{u}(\tau_a) = \mathbf{u}_a \quad (38)$$

Taking the Laplace transform of (37), using the initial conditions shown by (38) and solving for the Laplace domain displacement leads to the following result.

$$\mathbf{u}(s) = \frac{\left(s\mathbf{u}_a + \dot{\mathbf{u}}_a + \frac{1}{s}\ddot{\mathbf{u}}_a \right) m + \left(\mathbf{u}_a + \frac{1}{s}\dot{\mathbf{u}}_a \right) c + \left(\frac{1}{s}\mathbf{u}_a \right) k}{ms^2 + cs + k} \quad (39)$$

A comparison of (39) with (36) clearly shows that the former is simply the latter with $\mathbf{F}(s)$ specified and multiplied through the equation. The solution for $\mathbf{H}(s)$, given by (36), as expected, is present on the right of the equality on (39). From the form of (39) it can clearly be seen that the initial conditions can be treated as forcing functions. The terms in the numerator of (39) have been separated, parenthetically, by their associated passive element correspondence. For each element, we posit the view that the Laplace transform of the initial conditions can be represented in a line graph as a force source in parallel with the branch representing the impedance of the corresponding passive source.

For the mass element, the direction of the Laplace domain initial conditions force source is in the opposite direction (from the inertial node towards the mass-associated node). For mass elements, one may also be interested in accounting for net unbalanced externally applied loads. For each such load, we posit the view that the Laplace transform of the load can be represented as a force source, directed in the appropriate direction, in parallel with the branch representing the impedance of the corresponding mass valued node. The phrase appropriate direction can be interpreted by the following examples: acceleration would be directed away from the inertial node and braking would be directed towards the inertial node. The force sources for the mass element, irrespective of the number of separate sources employed, sum together due to the fact that they are parallel secondary to being along branches connected to the same nodal pair (i.e. the inertial node and the mass valued node). One may thusly redraw all such sources for a mass element as a single, equivalent, force source that is in parallel with the branch containing the impedance for the corresponding mass element.

For the spring and damper element shown in Figure 2, the initial conditions can also be represented as force sources. Each source is in parallel with the corresponding element.

These initial condition force sources should not be placed in parallel to a mass element unless the corresponding spring or impedance is in parallel with the mass element in question. If one were to relax the displacement constraint of $\mathbf{u}_b(t) = \mathbf{0}$, the spring and damper would remain parallel but would be in series with the mass rather than in parallel (the corresponding equation of motion would change due to the fact that one would have to include the displacement and velocity at (b) in the solution). For any number of spring and damper elements connected in parallel, we may reduce the system to the equivalent impedance (inverse of the sum of the inverse of the individual impedances) in parallel with a force generator (the sum of the force generators from the individual elements). When a force generator representing the Laplace transform of the initial conditions for a spring or damper element is connected in parallel to the element and with the element nodes consisting of a mass valued node and the ground reference node, the direction of the force generator is opposite of the direction of the force flow through the impedance. The reasoning for this is due to the proper signing of the initial conditions as can be seen in Figure 2. Figure 3 shows a redrawing of Figure 2 to show the force generators for each element.

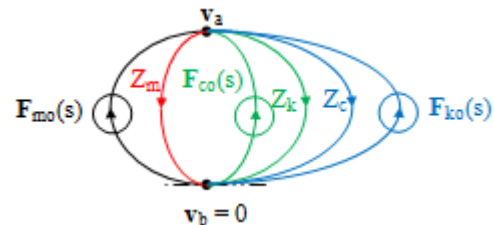


Figure 3: Line graph for the model shown in Figure 2 but with the initial conditions for each element shown as properly directed force generators for each element.

The correctness of this approach can readily be seen by summing the forces at node (a).

$$\begin{aligned} \mathbf{F}_{mo}(s) + \mathbf{F}_{co}(s) + \mathbf{F}_{ko}(s) &= \\ \mathbf{F}_m(s) + \mathbf{F}_c(s) + \mathbf{F}_k(s) &= \frac{\mathbf{v}_a}{Z_{eq}} \end{aligned} \quad (40)$$

In (40), the terms on the left side of the first equality are the three individual terms shown in the numerator of the term on the right of the equality of (39). It should be readily apparent that these equations can be used for modeling both the closure and separation phase of a collision for a SDOF model simply by changing the values of the initial conditions and the values of the stiffness and damping that are appropriate for the phase under consideration. This can be done without changing the form of the equations. This simple example covers the SDOF model for a vehicle to FRMB collision with the caveats noted above. In the following section we address the two body problem.

3. Applications

3.1 The general two body problem

In the Introduction to the subject work, the simple two body collinear collision problem was introduced along with the equation of motion for each body as per (1). The simple problem will be addressed, specifically, in the following

subsection. For now, we start the development of the general problem by redrawing Figure 1.

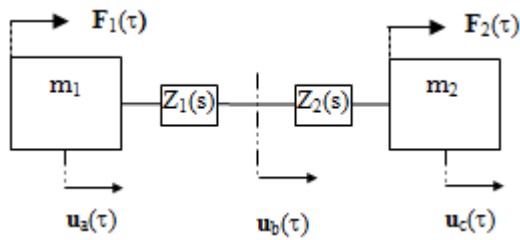


Figure 4: Schematic of the general two body problem showing the two masses, m_1 and m_2 , along with their respective displacements $\mathbf{u}_a(t)$ and $\mathbf{u}_b(t)$. The displacement $\mathbf{u}_b(t)$ represents a massless interface between the two objects. Each mass is connected to this interface by the appropriately subscripted impedance.

In Figure 4, the impedance $Z_1(s)$ denotes any number of springs and dampers connected between the first mass and the massless interface and $Z_2(s)$ denotes any number of springs and dampers connected between the massless interface and the second mass. This formulation, again, is used, because the impedances are treated as being characteristic of the relevant portion of the corresponding subscripted object. We first develop the transfer function for this system (under zero initial conditions) so that we can check the correctness of the solution developed after inclusion of the initial conditions as forcing functions. The line graph for this problem is shown Figure 5.

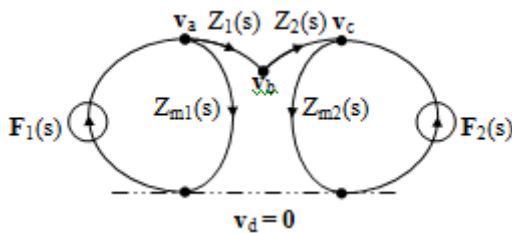


Figure 5: Line graph for the schematic shown in Fig. 4. The mass impedances and associated force generators are as labeled. The object associated (massless) impedances are shown as $Z_1(s)$ and $Z_2(s)$. The nodal velocities \mathbf{v}_a and \mathbf{v}_c correspond to the two masses. The nodal velocity \mathbf{v}_b is for the massless interface. The velocity \mathbf{v}_d is the reference velocity for the inertial frame (zero).

The sum of forces at node (a) leads to the following result.

$$\begin{aligned} \mathbf{F}_1(s) &= \mathbf{F}_{m_1}(s) + \mathbf{F}_{Z_1}(s) \\ &= \frac{1}{Z_{m_1}(s)}(\mathbf{v}_a(s) - \mathbf{v}_d(s)) \\ &\quad + \frac{1}{Z_1(s)}(\mathbf{v}_a(s) - \mathbf{v}_b(s)) \\ &= (\mathbf{Y}_{m_1}(s) + \mathbf{Y}_1(s))\mathbf{v}_a(s) - \mathbf{Y}_1(s)\mathbf{v}_b(s) \end{aligned} \tag{41}$$

The sum of forces at node (c) leads to the following result.

$$\begin{aligned} \mathbf{F}_2(s) &= \mathbf{F}_{m_2}(s) - \mathbf{F}_{Z_2}(s) \\ &= \frac{1}{Z_{m_2}(s)}(\mathbf{v}_c(s) - \mathbf{v}_d(s)) \\ &\quad - \frac{1}{Z_2(s)}(\mathbf{v}_b(s) - \mathbf{v}_c(s)) \\ &= (\mathbf{Y}_{m_2}(s) + \mathbf{Y}_2(s))\mathbf{v}_c(s) - \mathbf{Y}_2(s)\mathbf{v}_b(s) \end{aligned} \tag{42}$$

The sum of forces at node (b) leads to the following result.

$$\begin{aligned} \mathbf{F}_{Z_1}(s) &= \mathbf{F}_{Z_2}(s) \\ \frac{1}{Z_1(s)}(\mathbf{v}_a(s) - \mathbf{v}_b(s)) &= \frac{1}{Z_2(s)}(\mathbf{v}_b(s) - \mathbf{v}_c(s)) \\ \mathbf{v}_b(s) &= \frac{\mathbf{Y}_1(s)}{\mathbf{Y}_1(s) + \mathbf{Y}_2(s)}\mathbf{v}_a(s) \\ &\quad + \frac{\mathbf{Y}_2(s)}{\mathbf{Y}_1(s) + \mathbf{Y}_2(s)}\mathbf{v}_c(s) \end{aligned} \tag{43}$$

Substitution of (43) into (41) and (42) followed by algebraic simplification leads to the following results in which $\mathbf{v}_b(s)$ is eliminated from the equations.

$$\mathbf{F}_1(s) = (\mathbf{Y}_{m_1}(s) + \mathbf{Y}_{red}(s))\mathbf{v}_a(s) - \mathbf{Y}_{red}(s)\mathbf{v}_c(s) \tag{44}$$

$$\mathbf{F}_2(s) = (\mathbf{Y}_{m_2}(s) + \mathbf{Y}_{red}(s))\mathbf{v}_c(s) - \mathbf{Y}_{red}(s)\mathbf{v}_a(s) \tag{45}$$

Where $\mathbf{Y}_{red}(s)$ is the expected result of the equivalent admittance of the two admittances in series.

$$\mathbf{Y}_{red}(s) = \frac{\mathbf{Y}_1(s)\mathbf{Y}_2(s)}{\mathbf{Y}_1(s) + \mathbf{Y}_2(s)} \tag{46}$$

Equations (44) and (45) may be written in compact vector-matrix notation.

$$\begin{bmatrix} \mathbf{F}_1(s) \\ \mathbf{F}_2(s) \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{m_1}(s) + \mathbf{Y}_{red}(s) & -\mathbf{Y}_{red}(s) \\ -\mathbf{Y}_{red}(s) & \mathbf{Y}_{m_2}(s) + \mathbf{Y}_{red}(s) \end{bmatrix} \begin{bmatrix} \mathbf{v}_a(s) \\ \mathbf{v}_c(s) \end{bmatrix} \tag{47}$$

$$[\mathbf{F}(s)] = [\mathbf{Y}(s)][\mathbf{v}(s)]$$

Assuming that $[\mathbf{Y}(s)]$ is non-singular, the inversion of (47) leads to the following result (the explicit dependence on the Laplace variable is not shown for the admittances for the purpose of clarity).

$$[\mathbf{v}(s)] = \frac{\begin{bmatrix} \mathbf{Y}_{m_2} + \mathbf{Y}_{red} & \mathbf{Y}_{red} \\ \mathbf{Y}_{red} & \mathbf{Y}_{m_1} + \mathbf{Y}_{red} \end{bmatrix}}{\mathbf{Y}_{m_1}\mathbf{Y}_{m_2} + \mathbf{Y}_{red}(\mathbf{Y}_{m_1} + \mathbf{Y}_{m_2})} [\mathbf{F}(s)] \tag{48}$$

Using the relationship is $\mathbf{u}(s) = \mathbf{v}(s)/s$ leads to the following result.

$$[\mathbf{u}(s)] = \left(\frac{1}{s}\right) \frac{\begin{bmatrix} \mathbf{Y}_{m_2} + \mathbf{Y}_{red} & \mathbf{Y}_{red} \\ \mathbf{Y}_{red} & \mathbf{Y}_{m_1} + \mathbf{Y}_{red} \end{bmatrix}}{\mathbf{Y}_{m_1}\mathbf{Y}_{m_2} + \mathbf{Y}_{red}(\mathbf{Y}_{m_1} + \mathbf{Y}_{m_2})} [\mathbf{F}(s)] \tag{49}$$

In (49), the entirety of the term that pre-multiplies $[\mathbf{F}(s)]$ is the transfer function $H(s)$. This form of the transfer function holds when the system architecture is reducible to that shown in Figure 5. We now consider the case of initial conditions for this general case. For the mass valued nodes, the initial conditions can be shown as force generators, in parallel with the mass impedance branches, with opposite directionality,

as discussed previously. This is because the nodal connections for the masses are the same as they were for the previously discussed example. The previous statements regarding the additivity of additional forces, such as net unbalanced externally applied forces, still holds apt for this case. For the elements connected between the mass valued nodes and the massless interface node, we posit that the force generator for the initial conditions will be directed from the massless interface, towards the mass valued nodes. The resultant line graph is shown in Figure 6.

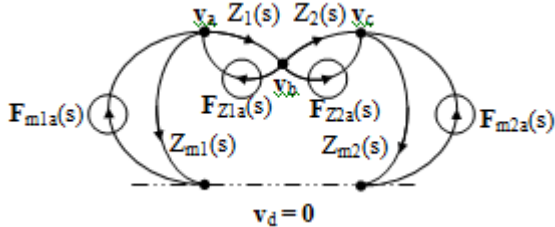


Figure 6: A modified version of Figure 5 showing the general forcing functions as being specifically due to the initial conditions present for each mass at $\tau = \tau_a$. Also shown are the branches for the force generators for the initial conditions at the same time for each impedance present along the branch from the mass valued nodes to the massless node.

The presence of the force generators for the impedances $Z_1(s)$ and $Z_2(s)$ clearly lead to an alteration of the sum of forces at each of the three salient nodes. This requires rewriting (41), (42) and (43). For this rewrite, we drop the depiction of the explicit dependance upon the Laplace variable, for purposes of clarity (the dependance still remains). Rewriting (41) leads to the following.

$$\begin{aligned} \mathbf{F}_{m_1a} &= \mathbf{F}_{m_1} + \mathbf{F}_{Z_1} - \mathbf{F}_{Z_{1a}} \\ &= \frac{1}{Z_{m_1}}(\mathbf{v}_a - \mathbf{v}_d) + \frac{1}{Z_1}(\mathbf{v}_a - \mathbf{v}_b) - \mathbf{F}_{Z_{1a}} \\ &= (\mathbf{Y}_{m_1} + \mathbf{Y}_1)\mathbf{v}_a - \mathbf{Y}_1\mathbf{v}_b - \mathbf{F}_{Z_{1a}} \end{aligned} \quad (50)$$

Rewriting (42) leads to the following.

$$\begin{aligned} \mathbf{F}_{m_2a} &= \mathbf{F}_{m_2} - \mathbf{F}_{Z_2} - \mathbf{F}_{Z_{2a}} \\ &= \frac{1}{Z_{m_2}}(\mathbf{v}_c - \mathbf{v}_d) - \frac{1}{Z_2}(\mathbf{v}_b - \mathbf{v}_c) - \mathbf{F}_{Z_{2a}} \\ &= (\mathbf{Y}_{m_2} + \mathbf{Y}_2)\mathbf{v}_c - \mathbf{Y}_2\mathbf{v}_b - \mathbf{F}_{Z_{2a}} \end{aligned} \quad (51)$$

Finally, rewriting (43) leads to the following.

$$\begin{aligned} \mathbf{F}_{Z_1} &= \mathbf{F}_{Z_2} + \mathbf{F}_{Z_{1a}} + \mathbf{F}_{Z_{2a}} \\ \frac{1}{Z_1}(\mathbf{v}_a - \mathbf{v}_b) &= \frac{1}{Z_2}(\mathbf{v}_b - \mathbf{v}_c) + \mathbf{F}_{Z_{1a}} + \mathbf{F}_{Z_{2a}} \end{aligned} \quad (52)$$

$$\mathbf{v}_b = \frac{\mathbf{Y}_1}{\mathbf{Y}_1 + \mathbf{Y}_2}\mathbf{v}_a + \frac{\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2}\mathbf{v}_c - \frac{\mathbf{F}_{Z_{1a}} + \mathbf{F}_{Z_{2a}}}{\mathbf{Y}_1 + \mathbf{Y}_2}$$

Substitution of (52) into (50) and (51) followed by algebraic rearrangement leads to the following results.

$$\begin{aligned} \mathbf{F}_{m_1a} + \frac{\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2}\mathbf{F}_{Z_{1a}} - \frac{\mathbf{Y}_1}{\mathbf{Y}_1 + \mathbf{Y}_2}\mathbf{F}_{Z_{2a}} &= \\ (\mathbf{Y}_{m_1} + \mathbf{Y}_{red})\mathbf{v}_a - \mathbf{Y}_{red}\mathbf{v}_c \end{aligned} \quad (53)$$

$$\begin{aligned} \mathbf{F}_{m_2a} - \frac{\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2}\mathbf{F}_{Z_{1a}} + \frac{\mathbf{Y}_1}{\mathbf{Y}_1 + \mathbf{Y}_2}\mathbf{F}_{Z_{2a}} &= \\ (\mathbf{Y}_{m_2} + \mathbf{Y}_{red})\mathbf{v}_c - \mathbf{Y}_{red}\mathbf{v}_a \end{aligned} \quad (54)$$

These two equations can again be written into a convenient vector-matrix form.

$$\begin{aligned} \begin{bmatrix} \mathbf{F}_{m_1a} \\ \mathbf{F}_{m_2a} \end{bmatrix} + \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} \begin{bmatrix} +\mathbf{Y}_2 & -\mathbf{Y}_1 \\ -\mathbf{Y}_2 & +\mathbf{Y}_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{Z_{1a}} \\ \mathbf{F}_{Z_{2a}} \end{bmatrix} &= \\ \begin{bmatrix} \mathbf{Y}_{m_1} + \mathbf{Y}_{red} & -\mathbf{Y}_{red} \\ -\mathbf{Y}_{red} & \mathbf{Y}_{m_2} + \mathbf{Y}_{red} \end{bmatrix} \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_c \end{bmatrix} \end{aligned} \quad (55)$$

Inverting the relationship as before leads to the following result.

$$\begin{aligned} \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_c \end{bmatrix} &= \frac{\begin{bmatrix} \mathbf{Y}_{m_2} + \mathbf{Y}_{red} & \mathbf{Y}_{red} \\ \mathbf{Y}_{red} & \mathbf{Y}_{m_1} + \mathbf{Y}_{red} \end{bmatrix}}{\mathbf{Y}_{m_1}\mathbf{Y}_{m_2} + \mathbf{Y}_{red}(\mathbf{Y}_{m_1} + \mathbf{Y}_{m_2})} \\ \cdot \left(\begin{bmatrix} \mathbf{F}_{m_1a} \\ \mathbf{F}_{m_2a} \end{bmatrix} + \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} \begin{bmatrix} +\mathbf{Y}_2 & -\mathbf{Y}_1 \\ -\mathbf{Y}_2 & +\mathbf{Y}_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{Z_{1a}} \\ \mathbf{F}_{Z_{2a}} \end{bmatrix} \right) \end{aligned} \quad (56)$$

Using the relationship of $\mathbf{u}(s) = \mathbf{v}(s)/s$ leads to the following result.

$$\begin{aligned} \begin{bmatrix} \mathbf{u}_a \\ \mathbf{u}_c \end{bmatrix} &= \left(\frac{1}{s} \right) \frac{\begin{bmatrix} \mathbf{Y}_{m_2} + \mathbf{Y}_{red} & \mathbf{Y}_{red} \\ \mathbf{Y}_{red} & \mathbf{Y}_{m_1} + \mathbf{Y}_{red} \end{bmatrix}}{\mathbf{Y}_{m_1}\mathbf{Y}_{m_2} + \mathbf{Y}_{red}(\mathbf{Y}_{m_1} + \mathbf{Y}_{m_2})} \\ \cdot \left(\begin{bmatrix} \mathbf{F}_{m_1a} \\ \mathbf{F}_{m_2a} \end{bmatrix} + \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} \begin{bmatrix} +\mathbf{Y}_2 & -\mathbf{Y}_1 \\ -\mathbf{Y}_2 & +\mathbf{Y}_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{Z_{1a}} \\ \mathbf{F}_{Z_{2a}} \end{bmatrix} \right) \end{aligned} \quad (57)$$

A comparison between (49) and (57) shows that the transfer function shown in the former is exactly the same as that generated in the latter. There are two differences between the two equations. The first, being minor, is that the generic forces $\mathbf{F}_1(s)$ and $\mathbf{F}_2(s)$ have been specified as being the Laplace domain transformed initial conditions that are operative on each mass. We again note that any net unbalanced externally applied force acting on either mass can be additively included within these terms as before. The second difference is that the second additive term in the terminus parenthetical of (57) consists of the Laplace domain transformed initial conditions at $\tau = \tau_a$ that are operative on the elements that are connected in series at the common massless interface. The validity of (57) can be checked by considering specific element compositions for which the solution is known.

3.2 Spring-spring connection

For the case in which the two elements connected in series at the massless interface are linear springs, one may readily use (1) to write the equations of motion for each mass at $\tau = \tau_a$ (we again note that the values for k_1 and k_2 are operative from τ_a forward; the double letters in the subscript refers to the point followed by the time).

$$\begin{aligned} m_1\ddot{\mathbf{u}}_a(\tau) &= m_1\ddot{\mathbf{u}}_{aa} - k_1(\delta_1(\tau) - \delta_{1a}) \\ m_2\ddot{\mathbf{u}}_c(\tau) &= m_2\ddot{\mathbf{u}}_{ca} + k_2(\delta_2(\tau) - \delta_{2a}) \end{aligned} \quad (58)$$

The force balance condition at the massless interface can be written as follows.

$$k_1 (\delta_1(\tau) - \delta_{1a}) = k_2 (\delta_2(\tau) - \delta_{2a}) \quad (59)$$

Substitution for the deflections in terms of the corresponding relative displacements in (58) and (59), algebraically solving (59) for the displacement $u_b(\tau)$, substitution of the resultant in the two equations under (58) and algebraically rearranging the results leads to the following time-domain form.

$$\begin{aligned} m_1 \ddot{u}_a(\tau) + k_{red} u_a(\tau) - k_{red} u_c(\tau) &= \\ m_1 \ddot{u}_{aa} + k_{red} (u_{aa} - u_{ca}) & \\ m_2 \ddot{u}_c(\tau) + k_{red} u_c(\tau) - k_{red} u_a(\tau) &= \\ m_2 \ddot{u}_{ca} - k_{red} (u_{aa} - u_{ca}) & \end{aligned} \quad (60)$$

In (60), the term k_{red} refers to the reduced stiffness and is equal to the product of k_1 and k_2 , divided by the sum of k_1 and k_2 . Both equations may be written as a single equation by employing vector-matrix notation.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_a(\tau) \\ \ddot{u}_c(\tau) \end{bmatrix} + \begin{bmatrix} +k_{red} & -k_{red} \\ -k_{red} & +k_{red} \end{bmatrix} \begin{bmatrix} u_a(\tau) \\ u_c(\tau) \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_{aa} \\ \ddot{u}_{ca} \end{bmatrix} + \begin{bmatrix} +k_{red} & -k_{red} \\ -k_{red} & +k_{red} \end{bmatrix} \begin{bmatrix} u_{aa} \\ u_{ca} \end{bmatrix} \quad (61)$$

Writing the matrix of masses as $[M]$, the matrix of stiffnesses as $[K]$ and the column vectors of the acceleration and displacements at $\tau = \tau_a$ in short form followed by taking the Laplace transform and rearranging leads to the following result.

$$[u(s)] = (s^2 [M] + [K])^{-1} \left(s [M][u_a] + [M][\dot{u}_a] + \frac{1}{s} ([M][\ddot{u}_a] + [K][u_a]) \right) \quad (62)$$

On the right side of the equality in (62), the first term is the transfer function $H(s)$ while the second term (within the second parenthetical) is the forcing function secondary to the initial conditions. Substitution of the masses and reduced stiffnesses into the matrices of $H(s)$ from (62) and expanding the result leads to the following.

$$H(s) = \frac{\begin{bmatrix} m_2 s^2 + k_{red} & k_{red} \\ k_{red} & m_1 s^2 + k_{red} \end{bmatrix}}{s^2 (m_{red} s^2 + k_{red})(m_1 + m_2)} \quad (63)$$

For this model, the impedances of the masses are $1/(sm_1)$ and $1/(sm_2)$ and the impedances of the stiffnesses are s/k_1 and s/k_2 . The admittances of the masses are sm_1 and sm_2 . The admittance of the springs are k_1/s and k_2/s . The product of the spring admittances is $(k_1 k_2)/s^2$. The sum of the spring admittances is $(k_1 + k_2)/s$. The reduced spring admittance is the product of the spring admittances divided by the sum of the spring admittances, which results in a value of k_{red}/s . Substitution of these results into the transfer function of (57) results in the following.

$$H(s) = \frac{\begin{bmatrix} m_2 s^2 + k_{red} & k_{red} \\ k_{red} & m_1 s^2 + k_{red} \end{bmatrix}}{s^2 (m_{red} s^2 + k_{red})(m_1 + m_2)} \quad (64)$$

It is clear that (63) and (64) match exactly. Expanding the initial conditions based forcing function from (62) leads to the following result.

$$\begin{bmatrix} sm_1 u_{aa} + m_1 \dot{u}_{aa} + \frac{1}{s} m_1 \ddot{u}_{aa} + \frac{1}{s} k_{red} (u_{aa} - u_{ca}) \\ sm_2 u_{ca} + m_2 \dot{u}_{ca} + \frac{1}{s} m_2 \ddot{u}_{ca} + \frac{1}{s} k_{red} (u_{ca} - u_{aa}) \end{bmatrix} \quad (65)$$

For (57), the initial conditions based forcing functions associated with the masses, by definition, are the following.

$$\begin{bmatrix} sm_1 u_{aa} + m_1 \dot{u}_{aa} + \frac{1}{s} m_1 \ddot{u}_{aa} \\ sm_2 u_{ca} + m_2 \dot{u}_{ca} + \frac{1}{s} m_2 \ddot{u}_{ca} \end{bmatrix} \quad (66)$$

Evaluation of the terms of the initial conditions based forcing functions associated with the stiffnesses leads to the following result.

$$\frac{1}{k_1 + k_2} \begin{bmatrix} +k_2 & -k_1 \\ -k_2 & +k_1 \end{bmatrix} \begin{bmatrix} +\frac{1}{s} k_1 \delta_{1a} \\ -\frac{1}{s} k_2 \delta_{2a} \end{bmatrix} = \frac{k_{red}}{s} \begin{bmatrix} u_{aa} - u_{ca} \\ u_{ca} - u_{aa} \end{bmatrix} \quad (67)$$

The sum of (66) and (67) exactly reproduces the initial conditions based forcing function of (65).

3.3 Damper-damper connection

A model consisting of two linear deflection rate dependent dampers is clearly not an appropriate closure phase model. However, it does provide another crystalized developable model against which the general solution can be tested. The underlying model development is presented in abbreviated form. The equations of motion for this case are as follows.

$$\begin{aligned} m_1 \ddot{u}_a(\tau) &= m_1 \ddot{u}_{aa} - c_1 (\dot{\delta}_1(\tau) - \dot{\delta}_{1a}) \\ m_2 \ddot{u}_c(\tau) &= m_2 \ddot{u}_{ca} + c_2 (\dot{\delta}_2(\tau) - \dot{\delta}_{2a}) \end{aligned} \quad (68)$$

The balance of forces at the massless interface can be written as follows.

$$c_1 (\dot{\delta}_1(\tau) - \dot{\delta}_{1a}) = c_2 (\dot{\delta}_2(\tau) - \dot{\delta}_{2a}) \quad (69)$$

Following the same derivation as with the previous instantiated case, the Laplace domain solution for the transfer function and initial conditions based forcing functions can be written as follows. The Laplace domain displacement solution is the vector-matrix multiplication of the latter by the former.

$$H(s) = (s^2 [M] + s [C])^{-1} = \frac{1}{s^2 (sm_{red} + c_{red})(m_1 + m_2)} \begin{bmatrix} m_2 s + c_{red} & c_{red} \\ c_{red} & m_1 s + c_{red} \end{bmatrix} \quad (70)$$

$$\begin{bmatrix} (s\mathbf{u}_{aa} + \dot{\mathbf{u}}_{aa} + s^{-1}\ddot{\mathbf{u}}_{aa})m_1 + c_{red} \left(\begin{matrix} (\mathbf{u}_{aa} - \mathbf{u}_{ca}) + \\ s^{-1}(\dot{\mathbf{u}}_{aa} - \dot{\mathbf{u}}_{ca}) \end{matrix} \right) \\ (s\mathbf{u}_{ca} + \dot{\mathbf{u}}_{ca} + s^{-1}\ddot{\mathbf{u}}_{ca})m_2 + c_{red} \left(\begin{matrix} (\mathbf{u}_{ca} - \mathbf{u}_{aa}) + \\ s^{-1}(\dot{\mathbf{u}}_{ca} - \dot{\mathbf{u}}_{aa}) \end{matrix} \right) \end{bmatrix} \quad (71)$$

In (70) and (71) the term c_{red} is the product of the damping coefficients divided by their sum. Turning to the general form of (57), we first consider the transfer function. The mass admittances, as before, are sm_1 and sm_2 . The impedances of the damping elements are $1/c_1$ and $1/c_2$ with corresponding admittances of c_1 and c_2 . The product of the damper admittances is c_1c_2 and the sum of the same is $c_1 + c_2$. Substitution of these values into the transfer function portion of (57) followed by algebraic reorganization leads to the following solution.

$$H(s) = \frac{1}{s^2(sm_{red} + c_{red})(m_1 + m_2)} \begin{bmatrix} m_2s + c_{red} & c_{red} \\ c_{red} & m_1s + c_{red} \end{bmatrix} \quad (72)$$

The solution shown in (72) exactly matches the solution shown in (70). For the initial conditions, (66) again provides the components that are associated with each mass. The terms associated with the dampers evaluates in the following manner.

$$\frac{1}{c_1 + c_2} \begin{bmatrix} +c_2 & -c_1 \\ -c_2 & +c_1 \end{bmatrix} \begin{bmatrix} +\frac{1}{s}c_1\dot{\delta}_{1a} + c_1\delta_{1a} \\ -\frac{1}{s}c_2\dot{\delta}_{2a} - c_2\delta_{2a} \end{bmatrix} = c_{red} \begin{bmatrix} s^{-1}(\dot{\mathbf{u}}_{aa} - \dot{\mathbf{u}}_{ca}) + (\mathbf{u}_{aa} - \mathbf{u}_{ca}) \\ s^{-1}(\dot{\mathbf{u}}_{ca} - \dot{\mathbf{u}}_{aa}) + (\mathbf{u}_{ca} - \mathbf{u}_{aa}) \end{bmatrix} \quad (73)$$

In (73), the first term to the right of the equality is simply c_{red} . The sum of (66) and (73) exactly match the expected result given by (71).

3.4 Spring-damper connection

This model, again, is clearly not appropriate for a closure phase model but does provide yet another scenario by which the general solution of (57) can be compared with a solution derived by other means. For this model, we arbitrarily assign the linear spring element to the first mass and the linear damper element to the second element. The underlying model development is again presented in abbreviated form. The equations of motion for this case are as follows.

$$\begin{aligned} m_1\ddot{\mathbf{u}}_a(\tau) &= m_1\ddot{\mathbf{u}}_{aa} - k_1(\delta_1(\tau) - \delta_{1a}) \\ m_2\ddot{\mathbf{u}}_c(\tau) &= m_2\ddot{\mathbf{u}}_{ca} + c_2(\dot{\delta}_2(\tau) - \dot{\delta}_{2a}) \end{aligned} \quad (74)$$

The balance of forces at the massless interface can be written as follows.

$$k_1(\delta_1(\tau) - \delta_{1a}) = c_2(\dot{\delta}_2(\tau) - \dot{\delta}_{2a}) \quad (75)$$

Unlike the previous two cases in which the massless interface kinematics could be eliminated in the time domain,

the continuity condition for this case, given by (75), contains both the displacement and velocity of the interface. The process of solving for the interface kinematics requires first substituting the relative nodal displacements for the deflections and the relative nodal velocities for the deflection rates, taking the Laplace transform and solving for the Laplace domain displacement $\mathbf{u}_b(s)$. This solution is then substituted into the Laplace transform of the two equations under (74) followed by solving as before. The transfer function for this case is the following.

$$H(s) = \begin{bmatrix} c_2m_2s^2 + k_1m_2s + c_2k_1 & c_2k_1 \\ c_2k_1 & c_2m_1s^2 + k_1m_1s + c_2k_1 \end{bmatrix} \frac{1}{s^2(c_2m_{red}s^2 + k_1m_{red}s + c_2k_1)(m_1 + m_2)} \quad (76)$$

The Laplace domain solution for the initial conditions based forcing function is the following.

$$\begin{aligned} [M](s[\mathbf{u}_a] + [\dot{\mathbf{u}}_a]) + s^{-1}[M][\ddot{\mathbf{u}}_a] + \\ \frac{c_2k_1}{sc_2 + k_1} \begin{bmatrix} +(\delta_{1a} + \delta_{2a}) + s^{-1}\dot{\delta}_{2a} \\ -(\delta_{1a} + \delta_{2a}) - s^{-1}\dot{\delta}_{2a} \end{bmatrix} \end{aligned} \quad (77)$$

For the general solution given by (57), the mass, damper and spring admittances are as before. The product of the damper and spring admittances is $(c_2k_1)/s$. The sum of the damper and spring admittances is $(sc_2 + k_1)/s$. The reduced admittance of the damper and spring admittances is $(c_2k_1)/(c_2s + k_1)$. The matrix term of the transfer function shown in (57) evaluates to the following.

$$\begin{bmatrix} m_2s + \frac{c_2k_1}{c_2s + k_1} & \frac{c_2k_1}{c_2s + k_1} \\ \frac{c_2k_1}{c_2s + k_1} & m_1s + \frac{c_2k_1}{c_2s + k_1} \end{bmatrix} \quad (78)$$

The term in the denominator of (57), excluding the parenthetical $1/s$ term, evaluates to the following.

$$m_1m_2s^2 + \frac{sc_2k_1(m_1 + m_2)}{c_2s + k_1} \quad (79)$$

Dividing (78) by (79) and multiplying the resultant by $1/s$ exactly reproduces the transfer function of (76). For the initial conditions, the terms associated with the mass are again given by (66). This leaves the terms associated with the damper and the spring. Substitution of the admittance terms into the second term of the parenthetical of (57) leads to the following result.

$$\begin{aligned} \frac{1}{sc_2 + k_1} \begin{bmatrix} +sc_2 & -k_1 \\ -sc_2 & +k_1 \end{bmatrix} \begin{bmatrix} +\frac{1}{s}k_1\delta_{1a} \\ -\frac{1}{s}c_2\dot{\delta}_{2a} - c_2\delta_{2a} \end{bmatrix} = \\ \begin{bmatrix} +\frac{c_2k_1}{sc_2 + k_1}(\delta_{1a} + \delta_{2a}) + \frac{c_2k_1}{s(sc_2 + k_1)}\dot{\delta}_{2a} \\ -\frac{c_2k_1}{sc_2 + k_1}(\delta_{1a} + \delta_{2a}) - \frac{c_2k_1}{s(sc_2 + k_1)}\dot{\delta}_{2a} \end{bmatrix} \end{aligned} \quad (80)$$

The solution given by (80) exactly matches the term following the second addition operation in (77).

3.5 Parallel spring-damper in series connection

This model, yet again, is inappropriate for the entirety of the closure phase but may provide utility when it comes to the evaluation of the separation phase. Regardless, the model is again presented in a general format as per the initial values, so that it can be utilized for any portion of either phase. The equations of motion for this model are given by the following.

$$\begin{aligned}
 m_1 \ddot{\mathbf{u}}_a(\tau) &= m_1 \ddot{\mathbf{u}}_{aa} - c_1 (\dot{\delta}_1(\tau) - \dot{\delta}_{1a}) \\
 &\quad - k_1 (\delta_1(\tau) - \delta_{1a}) \\
 m_2 \ddot{\mathbf{u}}_c(\tau) &= m_2 \ddot{\mathbf{u}}_{ca} + c_2 (\dot{\delta}_2(\tau) - \dot{\delta}_{2a}) \\
 &\quad + k_2 (\delta_2(\tau) - \delta_{2a})
 \end{aligned} \tag{81}$$

The balance of forces at the massless interface is the following.

$$\begin{aligned}
 c_1 (\dot{\delta}_1(\tau) - \dot{\delta}_{1a}) + k_1 (\delta_1(\tau) - \delta_{1a}) &= \\
 c_2 (\dot{\delta}_2(\tau) - \dot{\delta}_{2a}) + k_2 (\delta_2(\tau) - \delta_{2a}) &
 \end{aligned} \tag{82}$$

The displacement and velocity terms associated with the massless node are once again eliminated by first writing the deflections and their rates, in (81) and (82), in terms of the displacements and their rates, taking the Laplace transform of the resultants, solving for $\mathbf{u}_b(s)$ from the rewritten and transformed version of (82), and substituting the results in the two rewritten and transformed versions of the equations under (81). To simplify the presentation the term α is defined as follows. This definition should not be confused with any other usage of the symbol.

$$\alpha = \frac{(sc_1 + k_1)(sc_2 + k_2)}{s \sum c + \sum k} \tag{83}$$

The resultant transfer function is the following.

$$H(s) = \frac{\begin{bmatrix} s^2 m_2 + \alpha & \alpha \\ \alpha & s^2 m_1 + \alpha \end{bmatrix}}{s^2 (m_{red} s^2 + \alpha)(m_1 + m_2)} \tag{84}$$

The Laplace domain solution for the initial conditions based forcing function is the following.

$$\begin{aligned}
 [M] (s[\mathbf{u}_a] + [\dot{\mathbf{u}}_a] + s^{-1}[\ddot{\mathbf{u}}_a]) + \\
 \frac{c_1 c_2}{s \sum c + \sum k} \begin{bmatrix} \dot{\mathbf{u}}_{aa} - \dot{\mathbf{u}}_{ca} \\ \dot{\mathbf{u}}_{ca} - \dot{\mathbf{u}}_{aa} \end{bmatrix} + \\
 \frac{1}{s(\sum c + \sum k)} \begin{bmatrix} +c_1 k_2 \dot{\delta}_{1a} + c_2 k_1 \dot{\delta}_{2a} \\ -c_1 k_2 \dot{\delta}_{1a} - c_2 k_1 \dot{\delta}_{2a} \end{bmatrix} + \\
 \frac{(sc_1 + k_1)(sc_2 + k_2)}{s(\sum c + \sum k)} \begin{bmatrix} \mathbf{u}_a - \mathbf{u}_c \\ \mathbf{u}_c - \mathbf{u}_a \end{bmatrix}
 \end{aligned} \tag{85}$$

For the general solution given by (57), the admittances are as before. The admittance of each spring and damper in parallel is $(sc + k)/s$, where c and k carry the appropriate subscripts. The product of these admittances is $(sc_1 + k_1)(sc_2 + k_2)/s^2$. The sum of these admittances is $(s(c_1 + c_2) + (k_1 + k_2))/s$. The reduced admittance is $(sc_1 + k_1)(sc_2 + k_2)/(s(s(c_1$

+ $c_2) + (k_1 + k_2))$, which is equal to α/s . Substitution of this result along with the mass admittances and the $1/s$ term into the transfer function of (57) exactly reproduces the transfer function of (84). The component of the forcing function due to the initial conditions placed upon the masses is again given by (66). The component due to the initial conditions placed upon the springs and dampers, again using (57), is the following.

$$\frac{1}{s \sum c + \sum k} \begin{bmatrix} +sc_2 + k_2 & -sc_1 - k_1 \\ -sc_2 + k_2 & +sc_1 + k_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{Z_{1a}} \\ \mathbf{F}_{Z_{2a}} \end{bmatrix} \tag{86}$$

Where the following are the evaluated Laplace domain components of the forcing function vector.

$$\begin{bmatrix} \mathbf{F}_{Z_{1a}} \\ \mathbf{F}_{Z_{2a}} \end{bmatrix} = \begin{bmatrix} +\frac{1}{s} k_1 \delta_{1a} + \frac{1}{s} c_1 \dot{\delta}_{1a} + c_1 \delta_{1a} \\ -\frac{1}{s} k_2 \delta_{2a} - \frac{1}{s} c_2 \dot{\delta}_{2a} - c_2 \delta_{2a} \end{bmatrix} \tag{87}$$

The substitution of (87) into (86) followed by the addition of (66) results in the exact same solution given by (85).

4. Discussion

The subject presentation of lumped parameter modeling, for use in the modeling of mechanical translational systems, differs slightly, in certain aspects, from what one may consider a typical presentation within contexts such as vibration analysis. The characterization of the differences as being slight derives from the fact that the differences are contextual rather than substantive. The first difference is the consideration of non-zero valued initial acceleration for the system masses. This is readily evidenced in the subject work by the first term to the right of the equality in (11). The second difference is the development of a consistent formulation which allows for the determination of the differential equations of motion at any point in time during the system response. This is most concretely manifested by the treatment of the response, within the subject translational collision context, as being separable into a closure and separation phase. Clearly, such a formulation requires the inclusion of the first development as well as the inclusion of terms associated with the deflection and deflection rate. The approach taken in the formulation also allows for a change of model configuration as well as a change of model parameter values, both of which are generally not employed nor relevant in a typical presentation.

The four examples considered in the subject work were simple examples. The intention behind using simple examples was not one of limiting problem complexity but rather for the purpose of having readily developable models from which the solutions could be compared to the solutions generated by the methodology used in the subject work. Each of the four models considered were first developed by solving the force balance constraint equation, in the appropriate domain, for the relevant kinematic parameter associated with the massless collision interface and then substituting that solution into the two equations of motion. This was done in the time domain for the spring-spring and damper-damper cases. For the other two cases, the unilateral

Laplace transform was applied first, in order to change the time domain problem, which contained both displacement and its time derivative, to one that was reduced to the Laplace domain displacement. The procedural approach of converting all equations from the time domain to the Laplace domain prior to eliminating the common collision interface associated kinematic variable(s) from the equations of motion could readily have been used for all four cases. Another point that is worthy of note is that the four cases, after elimination of the intermediate kinematic variables, were solved, using vector-matrix relationships, and with the matrices being 2 x 2 and vectors being 2 x 1. In all cases, one could have readily skipped the substitution step and solved the three equations for each case using 3 x 3 matrices and 3 x 1 vectors. In such a case, the Laplace domain vector of displacements would be the column vector $\{\mathbf{u}_a(s), \mathbf{u}_b(s), \mathbf{u}_c(s)\}$. This approach was not used due to the fact that the resultant transfer function would have been 3 x 3 and because the form of the forcing function vector would have differed.

One aspect that may not readily be appreciated unless one goes through the derivational process is the time involved with deriving the equations starting with the equations of motion and the constraint equation. This is especially the case when the constraint equation is used to solve for the massless node kinematics, first, followed by substitution into the equations of motion. The use of symbolic mathematics software does provide for a substantive increase in speed when compared to a manual derivation but may require additional processing and time for reducing the form of the resultants.

The use of the impedance analogy or admittance analogy, coupled with the appropriate relationships for parallel and series connections, increases the rapidity at which one may develop the Laplace domain transfer function for the underlying system. The force-current analogy was employed, in the subject work, to allow for the use of line graphs, which can readily be generated for mapping the system connectivity in terms of nodal velocities and force flows. The line graphs require a presumption of the direction of the nodal velocity drops, however, the specific direction chosen is immaterial to the degree that consistency is retained. In this regard, the requirement is no different than the consistency required with a free body diagram after defining a sign convention. It should be noted again, that the relationships for the force-voltage analogy differ from the force-current analogy and care should be taken to establish the analogy being used when considering any given source.

The primary contribution, to the literature, of the subject work, extends beyond the development of a method for calculating the transfer function. Rather, it is the development of a method for the contemporaneous determination of both the transfer function and the form of the forcing function vector due to the initial conditions present. For the force-current analogy, which differs from the force-voltage analogy [12], the initial conditions for each passive element in the system can be represented by a force generator in parallel with the element. For each mass element, the force generator that represents the initial

conditions must have one branch connected to the inertial node and with the other branch connected to the same node as the mass element. The most general form of the Laplace domain expressed forcing function of the j^{th} mass, connected to the p^{th} node (the non-reference node), referenced to time $\tau = \tau_a$, is given by the following.

$$F_{m_j,a}(s) = m_j (\mathbf{s}\mathbf{u}_{pa} + \dot{\mathbf{u}}_{pa} + s^{-1}\ddot{\mathbf{u}}_{pa}) \quad \forall j, p \quad (88)$$

Clearly, in (88), one may readily assign a zero valuation, as appropriate, to any of the kinematic terms. Net unbalanced externally applied forces, for which a Laplace transform can be defined, acting on a given mass, can also be applied as force generators that are in parallel with the mass. For the linear spring and linear damper elements, we again note that the initial conditions can be represented by a force generator that is in parallel with the underlying element. When the underlying elements do not have at least one branch connected to a mass associated system node, the admittances of the elements can be combined, using the parallel and series relationships for the force-current analogy. Care must be taken when considering multiple force sources along a single branch due to the fact that the force, analogous to current here, cannot have multiple values within a single branch. Force sources, in parallel, on the other hand, are additive for the subject analogy.

When a branch, containing either a linear spring or damper, is connected to a mass associated node and a massless interface node (e.g. Figure 6), the direction of the associated force course for the initial conditions is directed from the massless interface node and towards the mass associated node. This finding is dependent upon the sign convention used in this work in regards to the deflection and deflection rates and as a result upon the sign convention used in the line diagrams. This sign convention, described using terminology that is more befitting of common parlance rather than technical succinctness, is one of left to right. As an example, the deflection $\delta_i(\tau)$ was defined as $\mathbf{u}_a(\tau) - \mathbf{u}_b(\tau)$. The former is to the left of the latter as shown in Figure 4. This is then manifested as the corresponding velocity drop $\mathbf{v}_a(s) - \mathbf{v}_b(s)$ as shown in Figure 6. This finding is also manifested in the derivation of (57), which was the crowning finding of the subject work. If one considers the branch between nodes p (left) and q (right), where one node is mass associated and the other is not, the Laplace domain forcing function associated with a linear spring (with a stiffness of k_{pq} for $\tau \geq \tau_a$ and with an initial deflection of δ_{pqa} at $\tau = \tau_a$) is given by the following.

$$F_Z(s) = \pm s^{-1} k_{pq} \delta_{pqa} \quad (89)$$

The sign in (89) is based upon the aforementioned sign convention. Similarly, for a linear damper, the Laplace domain forcing function is given by the following.

$$F_Z(s) = \pm (s^{-1} c_{pq} \dot{\delta}_{pqa} + c_{pq} \delta_{pqa}) \quad (90)$$

A benefit of this approach is that the forms of (89) and (90) are consistent with the forms of the Laplace transform of the initial conditions corresponding to the linear spring and

linear damper, respectively. Equation (57) can readily produce the correct result for combined impedances, as shown by the fourth case that was developed above. Two caveats, specific to this point, however, are extremely relevant. The first is that the system architecture must consist of only two degrees of freedom, arranged as shown above, and must include a single massless, series, interface. As noted previously, (57) can readily be used with any combination of linear springs and dampers that connect a mass valued node to the massless interface node (by reduction to an effective admittance coupled with the appropriate single forcing function). The addition of additional mass valued nodes to the system would clearly change the form of (57).

The mention of the inclusion of additional mass associated nodes (i.e. additional degrees of freedoms) serves as an excellent segue for the discussion regarding the limitations of the subject approach and potential considerations for future developments. The inclusion of additional system mass associated nodes would clearly require the inclusion of a number of equations equal to the number of mass associated nodes added and a rewriting of the extant equations to the extent that each, individually, is impacted by the additional nodes. This holds for the subject admittance analogy approach as well as for the approaches based upon the Laplace transform of the coupled differential equations of motion. For a given topology of mass associated nodes, however, one may readily derive a generalized solution, based upon effective admittance, rather than having to derive the response for each specific case, based upon the arrangement of linear spring and damper elements, as one must do when working with the transformed equations of motion.

Both approaches do share a pair of limitations, equally applicable, due to their closed-form analytic nature. The first is that the models are uniaxial. The second is that both modelling approaches exhibit a degree of difficulty, in regards to implementation, as model complexity increases. A large portion of this derives from the fact that the underlying equations are formulated to allow for differentially valued model parameters based upon whether or not the loading, at any given time, for any given modeled load path, is compressive or tensile with respect to the dynamic reference state of the modeled load path (i.e. the deflection rate at $\tau = \tau_a$).

For the simple two mass problem considered in this work, if the collision is modeled using two phases (closure and separation), the dynamic reference state of the three nodes need only be known at the start of the closure phase and at the terminus of the closure phase (i.e. the start of the separation phase) to fully cover the necessary reference state values for modeling the entire collision.

A slightly more complex model is that of a three mass uniaxial collision model for the case of a frontal, inline, full-width engagement, impact between a test vehicle and a FRMB. The three masses, for such a case, consist of a modeled engine mass, a modeled front wheels and suspension mass and a modelled mass for the remaining

aspects of the test vehicle. This modeling approach derives from the fact that these aspects of any given test vehicle are instrumented with accelerometer blocks that have one sensing axis oriented along the longitudinal axis of the test vehicle, in its reference state.

For a fully connected model, there would be a total of six pathways that must be considered. Three of these are connections between the masses and the other three are from the connection of each mass to the barrier (treated as a fixed constraint). For such a case one may readily define closure and separation based upon the dynamic center of mass response, but each modeled load path can readily undergo multiple transitions from one state to the other. The implementation issue is eased with the used of an appropriate, approximate, numerical methods based approach, but the trade-off is the loss of a closed-form solution. A third limitation, within the same vein, is that the closed form analytic method is limited to formulations and functions for which the Laplace transform are determinable.

With these limitations in mind, one can still posit the view that there exist additional developments that can be made and with the subject work serving as a foundation. The first is the development of the framework, resulting in relationships corresponding to (57), for other commonly used model configurations. The second is the extension of the model for planar collision analysis. Finally, the third is the consideration of other force-deflection and force-deflection rate relationships beyond the linear case.

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Author Profile



Jai Singh obtained a B.S. degree in mechanical engineering from the University of Illinois at Urbana-Champaign and a M.S. degree in biomedical engineering from the University of Southern California. He is an accredited traffic accident reconstructionist and a biomechanical engineer. He has published numerous peer-reviewed papers, technical articles and conference proceeding papers. He has served as a peer paper reviewer. As the principle at Biomechanical Engineering Analysis and Research, Inc. he has evaluated thousands of real world accident cases.