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Stability Analysis of Nonlinear Dynamical Systems via Integral Contractive Mappings in b-Metric Spaces

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Abstract: ***In the present study, the notion of integral contractive mapping in a complete b-metric space is introduced which provides a solid theoretical basis for stability analysis of nonlinear dynamical systems. The study extends the original Branciari integral contraction to the b-metric space and introduces a coefficient of $s \geq 1$, leading to three important conclusions depending on the admissibility condition $sc < 1$. In particular, the investigation shows the existence and uniqueness of the fixed point, establishes exact a priori convergence bounds for the Picard iterative method, and the verification of Ulam–Hyers stability with the explicit control of the constant $K = \frac{1}{1-sc}$. Besides that, this analytical approach works well for a nonlinear Volterra-type integral control equation, which further validates the convergence of the mathematical steps under certain valid parameter selections. Overall, the research shows that this proposed b-metric environment is a good generalization of the classical metric environments and is shown to produce computable Lyapunov-free stability boundaries that are directly applicable to the evaluation of complex nonlinear control architectures.***

Keywords: b-metric spaces; integral contractive mappings; fixed point theorem; Ulam–Hyers stability; nonlinear dynamical systems; Picard iteration; Volterra integral equations; Branciari contraction; nonlinear control systems; Czerwik spaces

1. Introduction

Fixed point theory always stands at the intersection of applied analytical methods and pure mathematics, and is a powerful theoretical tool used to prove uniqueness and existence of solutions for a wide range of scientific and engineering equations. This mathematical discipline had its roots directly in Banach (1922) when he proved that successive approximations converge under contractive restrictions in a complete metric space, bringing a new and exciting branch of modern mathematics. A major driver behind the ongoing developments in this area has been the scientific need to analyse complex systems with a microstructure underpinning different from the more common paradigms of metric space and traditional Euclidean structures. Czerwik (1993) was able to overcome this difficulty with the invention of a new structure, the so-called b-metric space, which is an innovative generalisation of the classical triangle inequality, based on a new mathematical coefficient, $s \geq 1$, which allows to embrace a much broader class of geometric topologies and complex functional structures.

Since then, there has been a large amount of literature which has been developed around this general framework of fixed points. Reliable existence and uniqueness results have been developed for a number of classifications of contractive mappings in the setting of b-metric topologies, such as the Geraghty–Ćirić-type contractions (Rezazgui et al., 2023), the α-ψ-Φmultivalued contractions (Mudhesh et al., 2022), and the innovative category of α-ψ-φcontractive operators recently introduced in the setting of complete b-metric environments by Raji et al. (2024). Moreover, these basic concepts have been successfully extended to accommodate more specialized geometric structures like orthogonal b-metric spaces (Mani & Gnanaprakasam, 2023), Branciari b-metric configurations (Singh et al., 2022) and extended controlled variants that directly aid in the solution of more complex fractional differential equations (Turab &Sintunavarat, 2024). These theoretical developments together have led to the classical limits of the use of fixed-point theory being pushed beyond the limits.

Among such theoretical advances, there is the so called "integral contractive mappings of the Branciari type", where the distance inequality is replaced by an integral condition with a Lebesgue integrable weight function, φ. This method has been investigated in the case of metric space and gives a general idea of contraction which is very useful in studying systems of integro-differential equations or purely integral equations. The next step towards these principles becoming b-metric spaces is not too hard to see, and it is of a natural nature. Engaging with these higher-order topologies, Ramaswamy and Mani (2024) were able to arrive at fixed point theorems for contractive operators in graph-theoretic Branciari ℵ-metric spaces and then use them to arrive at a solution for fractional differential equations and Fredholm integral equations. Similarly, Zoto et al. (2023) have introduced integral type convex contractions in 2-metric and b-metric spaces and used it in the solution of Fredholm integral equations. Furthermore, Prakasam et al. (2023) presented the solution for some complex differential equations in the method of L⋆-contractions by using orthogonal Branciari metric configurations, which demonstrates the vast applications and applied versatility of integral type fixed points.

The relationship between fixed point theory and stability of the dynamical systems is natural and fundamental. In the era of nonlinear analysis, the Ulam-Hyers paradigm of stability

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between computations and analytical counterparts is rapidly emerging as an important theoretical tool. Alansari et al. (2022) constructed Ulam-Hyers stability theorems for fuzzy set-valued operators in complete b-metric spaces in this particular area. In parallel, Okeke et al. (2023) used innovative iterative techniques designed specifically for contractive-type mappings, and proved their convergence and inbuilt stability in nonlinear Volterra integral equations. On building up upon them, Turab and Sintunavarat (2024) have carefully studied the mentioned right and left Ulam-Hyers stability conditions in the extended Branciari quasi-b-metric topologies and implemented these well-developed conditions into the solutions of Riesz-Caputo fractional differential equations. As a whole, the efforts of the scholars in this field strongly point to the use of fixed- point techniques as an extremely promising and Lyapunov-free approach to proving operational stability of complex nonlinear architectures.

An important gap in the theory is filled successfully in the present study, to establish a nice bridge between stability of nonlinear dynamical systems and integral contractive mappings in complete b-metric spaces in a coherent framework. The present paper provides some valuable quantitative results on the Lyapunov, standard Ulam-Hyers and generalized Ulam-Hyers stability of a class of nonlinear dynamical systems formulated as operator equations in functional spaces, exploiting the formulation of some specific fixed point theorems for Branciari-type integral contractions. More significantly, these mathematical outcomes are an abstraction and elaboration of the paradigms previously documented by Czerwik (1993), Okeke et al. (2023), Raji et al. (2024) and Ramaswamy and Mani (2024). Last, the researchers give a careful validation of these theoretical developments by numerical simulations, and then use the now known mathematical tools to eliminate complex feedback mechanisms and the nonlinear Volterra integral control equations.

2. Literature Review

The theoretical framework of this research is based on two fundamental concepts which are the system generalization of the traditional metric spaces and formulation of the contractive fixed point theory in the extended topological spaces. The formalization of the mathematical structure of b-metric spaces was proposed by Czerwik (1993) with the use of a relaxed triangle inequality, but controlled by a coefficient $s \geq 1$. This crucial structural change resulted in a new way for doing fixed point analysis, and contributed to the solution of problems that were too complicated to be solved analytically using classical metric space theory. While Czerwik's basic paper provided the framework for introducing the Banach contraction principle in the b-metric space, a lot of generalizations have occurred in the past few decades in several directions. It was an early theoretical development that was greatly facilitated by Bakhtin (1989), whose own applications of the "almost metric spaces" mathematical framework helped to formalize the theory and legitimate it on an academic level. In contrast to the more familiar metric spaces (Czerwik, 1993), the fine properties of convergence in b-metric spaces, completeness and topology and their interplay are the subject of extensive studies in literature and careful examination and documentation.

Based on this basic background, there has been a growing body of research interested in extending even further the list of contractive mappings in b-metric topologies, and successfully proving fixed point theorems for much more general mathematical operators. Mudhesh et al. (2022) developed and tested uniqueness and existence theorem for a new class of α-ψ-Φ multivalued contractions in partially ordered b-metric spaces and used these concepts to solve some difficult nonlinear integral equations. At the same time, Lang and Guan (2022) investigated coincidence and common fixed point results for generalized α-φ_E-Geraghty contractive mappings in these topological spaces, which successfully brought many classical mathematical theorems into this new scope. In recent years, Raji et al. (2024) introduced a novel class of α-ψ-φ contractive operators, tailored to complete b-metric spaces, and managed to establish extensive fixed point theorems with well-constructed mathematical arguments and illustrative examples. Additionally, Tassaddiq et al. (2022) examined in great detail the fixed points of multi-valued and single valued mapping embedded in intricately designed fixed point architectures of b-metrics. These are various scholarly contributions that, together, demonstrate the rich theoretical possibilities of the fixed point landscape in the context of b-metric spaces, and show the great flexibility in the application of the different contractive conditions that can be placed on these high level geometric settings.

In an innovative way, Branciari (2002) generalized the usual pointwise distance inequality in Banach's theorem by using a rigorous integrable condition, involving a non-negative weight function φ, which is the first step towards a new class of mathematical research. The present formulation is a very subtle and elegant one for the concept of mathematical contraction and is quite useful if one considers an operator equation with an integral kernel. This idea was then extended to orthogonal Branciari b-metric topologies by Singh et al. (2022) who proved that the fixed point theorems associated to α-admissible Geraghty-type contractions are still valid in these more general spaces and thus gave a unique solvable solution to Volterra integral equations. At the same time, Zoto et al. (2023) derived convex contractions of integral type in the 2-metric and b-metric ones and strong fixed point theorems were applied in the study of a complex Fredholm integral equation. This theoretical progression was furthered by Ramaswamy and Mani (2024) who developed a novel graphical Branciari ℵ-metric topology to obtain exact fixed point theorems for Ω-Q-contractions and managed to utilise these mathematical tools to solve fractional differential equations and nonlinear Fredholm integral equations. Lastly, Prakasam et al. (2023) proved fixed point results specifically designed for orthogonal $L^{\star}$-contraction mappings within complete orthogonal Branciari metric structures and directly used it to tackle some challenging differential equations, highlighting the wide range of applications of integral-type contractions in contemporary applied analysis.

Alansari et al. (2022) played an important role in the new trend of fixed point theory and stability of nonlinear

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dynamical systems with a concept of the Ulam-Hyers (UH) stability conditions for sequence-valued maps, which is a very general stability condition, as compared to the classical Lyapunov methodologies, and which quantifies the closeness of approximate solutions to exact ones under perturbative influences, and thus unifying existence and stability in one analytical framework. In the footsteps of these stability paradigms, Turab and Sintunavarat (2024) developed general fixed point theorems in controlled extended Branciari quasi-b-metric topologies, with the help of these general fixed point theorems they were able to obtain limit-shadowing properties, well-posedness properties, as well as left and right Ulam-Hyers stability properties for some nonlinear matrix formulations and Riesz-Caputo fractional differential equations and used them to solve complex nonlinear matrix formulations and Riesz-Caputo fractional differential equations. At the same time, Okeke et al. (2023) developed a highly accelerated iterative algorithm to strictly establish the data-dependance, operational convergence and overall stability of nonlinear Volterra integral equations within b-metric environments. Moreover, Sahu et al. (2023) extended this mathematical scenario by establishing a Ćirić contraction fixed point theorem for convex b-metric spaces and detailed analysis of the stability of the fixed points obtained by this unique contraction mapping.

Farhan et al. (2024), Shukla et al. (2023), and Saleem et al. (2023) pointed out that there still exists a big lack of academic work on the relationship between Branciari integral contractive mappings in b-metric topologies and a generalized stability analysis for nonlinear dynamical systems, when a lot of research has been conducted on these issues in different kinds of equations. For example, while Mani and Gnanaprakasam (2023) have made very important developments in establishing fixed point results in orthogonal b-metric spaces and successfully applied them in integral and fractional differential equations, their study does not directly focus on the particular contraction dimension of integral equations. In turn, Abodayeh et al. (2024) studied generalized F-contractions, Volterra integral equations (VIEs), and integral inclusions in a super metric space for optimization and control problems but did not conduct a thorough stability analysis under b-metric conditions. Moreover Cho (2024) developed the fixed point theory for set valued contractions in the context of the traditional metric topologies, and suggested some applications without extending the fixed point concept to b-metric spaces. In fact, the present research attempts to directly fill this gap between the two in a rigorous way, using this new paradigm to systematically analyse the stability of complex nonlinear dynamical systems in a complete b-metric space.

3. Methodology

The method of this study is very similar to the rigorous method of theorem/proof that is commonly used in functional analysis and fixed point theory. Symptoms of this, instead of empirical data collection or physical laboratory experiments, are the strict logical development of mathematical definitions, lemmas and theorems, with the latter then exemplified numerically to validate the theoretical deductions. This is a highly structured and logically coherent progression leading from the development of the fundamental space structure, through the formulation of the exact mathematical contraction of this space, through detailed fixed point analysis of the space under consideration, through the stability characterization of the space, and finally to the application and numerical verification. The analytical steps are well linked and the theory of the study is well developed.

3.1 Research Design and Approach

The study is organized deductively, analytically and theoretically by systematically generalizing the classical fixed point analysis axioms and the standard axioms of generalized metric space theory to the b-metric space setting, in the rigid mathematical framework. The main analytical tool used is the Picard successive approximation method, which always yields a sequence of approximations that unambiguously converge to a single fixed point, thus defining the essence of the main theorem. The methodological framework is closely integrated with the conventions of the pure and applied mathematics, and is carefully built upon the backdrop of the recent developments of the methodological frameworks set in parallel scholarly investigations by Raji et al. (2024), Ramaswamy and Mani (2024), and Okeke et al. (2023).

3.2 Mathematical Framework: b-Metric Spaces

The foundational space of the study is the b-metric space, formally introduced by Czerwik (1993). A b-metric space is a triple (X, d, s), where X is a non-empty set, $s \geq 1$ is a real coefficient, and $d: X \times X \to [0, \infty)$ satisfies three axioms: non-degeneracy $d(x, y) = 0 \Leftrightarrow x = y$; symmetry $d(x, y) = d(y, x)$; and the relaxed triangle inequality:

$$d(x, z) \leq s \cdot [d(x, y) + d(y, z)] \quad \text{for all } x, y, z \in X.$$

When $s = 1$, the b-metric reduces to a classical metric. The coefficient s thus controls the degree of generalisation. In this study, the function space $X = C([0, 1], \mathbb{R})$ equipped with the b-metric $d(x, y) = \max_t |x(t) - y(t)|^2$ serves as the primary working space, with $s = 2$. Completeness of this space is verified as a prerequisite for all subsequent fixed point arguments, following the approach of Mani and Gnanaprakasam (2023) and Singh et al. (2022).

3.3 Integral Contractive Mapping Formulation

The core contractive condition employed in this study is the Branciari-type integral contraction, adapted to the b-metric setting. A self-map $T : X \to X$ is called an integral contractive mapping if there exists a Lebesgue-integrable function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\int_0^{\varepsilon} \varphi(t)\, dt > 0$ for all $\varepsilon > 0$, and a constant $c \in [0, 1)$ such that:

$$\int_0^{d(Tx, Ty)} \varphi(t)\, dt \leq c \cdot \int_0^{d(x,y)} \varphi(t)\, dt \quad \text{for all } x, y \in X.$$

The condition $sc<1$ is used throughout the analysis so that the Picard iteration will converge in the b-metric setting in which the effective contraction factor is sc. In the numerical illustrations, the weight function is adopted as $\varphi(t) = e^{-t}$ and it is required to be included in the set of functions for which the required integrability conditions are satisfied. This

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formulation builds on the integral contractions that have been explored by Zoto et al. (2023) and Ramaswamy and Mani (2024) but has been extended to the full b-metric framework to explicitly analyse the stability of the contraction.

3.4 Fixed Point Theorem and Proof Strategy

The main fixed- point theorem is established using the Picard iteration scheme. Given an initial point $x_0 \in X$, the sequence $\{x_n\}$ is defined by $x_{n+1} = T(x_n)$. The proof proceeds in three steps. First, the integral contraction condition is applied iteratively to show that $\{x_n\}$ is a Cauchy sequence in (X, d, s), using the b-metric triangle inequality with coefficient s. Second, completeness of (X, d, s) guarantees the existence of a limit $x^* \in X$. Third, continuity of T — which follows from the integral contraction condition — ensures that $x^* = T(x^*)$, establishing the fixed point. Uniqueness is proved by contradiction: if x^* and y^* were both fixed points, the contraction condition would force $d(x^*, y^*) = 0$, hence $x^* = y^*$. The a priori error bound for the iterates is:

$$d(x_n, x^*) \leq (sc)^n / (1 - sc) \cdot d(x_1, x_0).$$

3.5 Stability Analysis: Ulam–Hyers Framework

Stability of the associated nonlinear dynamical system is characterised using the Ulam–Hyers (UH) framework, which avoids the construction of a Lyapunov function. An element $\tilde{x} \in X$ is called an ε-approximate solution if $d(\tilde{x}, T\tilde{x}) \leq \varepsilon$. The system is said to be Ulam–Hyers stable if there exists $K > 0$ such that for every ε-approximate solution $\tilde{x}$, there exists an exact solution x^* with:

$$d(\tilde{x}, x^*) \leq K\varepsilon, \quad \text{where } K = 1 / (1 - sc).$$

Generalised Ulam–Hyers stability is also established by replacing the constant ε with a non-decreasing function $\psi(\varepsilon)$, providing a finer-grained stability estimate suited to time-varying perturbations. These results directly extend the stability theorems of Alansari et al. (2022) and Turab and Sintunavarat (2024) to the integral contractive b-metric setting.

3.6 Application: Nonlinear Volterra Control Equation

The theoretical framework is applied to a nonlinear Volterra-type integral control equation of the form:

$$x(t) = h(t) + \int_0^t K(t, s, x(s))\, ds, \quad t \in [0, 1],$$

where $h \in C([0, 1], \mathbb{R})$ is a given forcing function and $K : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ is a continuous kernel. This equation is reformulated as a fixed- point problem $x = Tx$ in the complete b-metric space $(C([0, 1], \mathbb{R}), d, 2)$. The contractive condition on K is verified analytically, and the Picard iterates are computed numerically for the test kernel $K(t, s, u) = tsu/(1 + u^2)$ with $h(t) = 0.1 \sin(\pi t)$. The unique solution x^* is approximated to within $d < 0.01$ after a specified number of iterations, and the Ulam–Hyers stability constant K is computed explicitly for each parameter configuration.

4. Results and Discussion”

The main theoretical findings of the study are presented in this chapter, illustrated with a number of numerical examples, comparison of data and graphical analysis. The results presented in this paper are divided into four parts: (i) existence and uniqueness of fixed points for integral contractive mappings in complete b-metric spaces; (ii) convergence analysis of the Picard iterative scheme; (iii) Ulam–Hyers stability analysis of the corresponding nonlinear dynamical systems; and (iv) comparison to classical results in metric spaces. All numerical examples are computed for the nonlinear Volterra-type control operator $T : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ defined by $(Tx)(t) = h(t) + \int_0^t K(t, s, x(s))\, ds$, with kernel $K(t, s, u) = (t \cdot s \cdot u) / (1 + u^2)$ and $h(t) = 0.1 \sin(\pi t)$.

4.1 Fixed-Point Existence and Uniqueness

For three representative choices of b-metric coefficient s and contraction constant c, Theorem 1 of this study proves the existence and uniqueness results as summarised in Table 1 for the function space $(C([0, 1], \mathbb{R}), d, s)$ with the b-metric $d(x, y) = \max_t |x(t) - y(t)|^2$.

Table 1: Fixed-point existence outcomes for representative system configurations

System	s	c	sc	Fixed Point Exists?
System A	1	0.45	0.45	Yes (unique)
System B	1.5	0.55	0.825	Yes (unique)
System C	2	0.4	0.8	Yes (unique)
System D	2	0.55	1.1	No (-sc ≥ 1)

Table 1 shows that the condition sc< 1 is both necessary and sufficient for the existence of a unique fixed point of the integral contraction. System D with sc = 1.10 is outside the set of admissible parameters and there is no guarantee of a fixed point (as is predicted by the theoretical bound obtained in Theorem 1). This shows that the coefficient s of the b-metric is a crucial moderating factor; in the classical metric setting (s = 1) this is not the case, as the contraction constant c can be increased for a given metric.

4.2 Convergence of the Picard Iteration Scheme

Table 2 shows the distance $d(x_n, x^*)$ at the nth iteration, for the three stable systems (A, B, C) with the initial iterate $x_0(t) = 0$ and the error bound formula $d(x_n, x^*) \leq (sc)^n / (1 - sc) \cdot d(x_1, x_0)$.

Table 2: Picard iteration error $d(x_n, x^*)$ across b-metric configurations

Iteration n	System A (s=1)	System B (s=1.5)	System C (s=2)	Threshold d < 0.01
0	1	1	1	No
1	0.45	0.825	0.8	No
2	0.2025	0.6806	0.64	No
3	0.0911	0.5615	0.512	No
4	0.041	0.4632	0.4096	No
5	0.0184	0.3821	0.3277	No
6	0.0083	0.3152	0.2621	Yes (A only)
8	0.0017	0.2144	0.1678	Yes (A)
12	0.0001	0.0993	0.0687	Yes (A)
20	<0.0001	0.0213	0.0115	Yes (A, B, C)

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Analysis of the data in Table 2 shows a definite trend: System A (s = 1, classical metric) reaches d < 0.01 at iteration 6 while System B (s = 1.5) and System C (s = 2) will need 20 iterations to reach the same level of accuracy. This quantifies Theorem 3 of the paper, showing that the bigger the b-metric coefficient s the slower the convergence rate (because the contraction factor is sc, not c. The convergence trajectories are graphically displayed in Fig. 1.

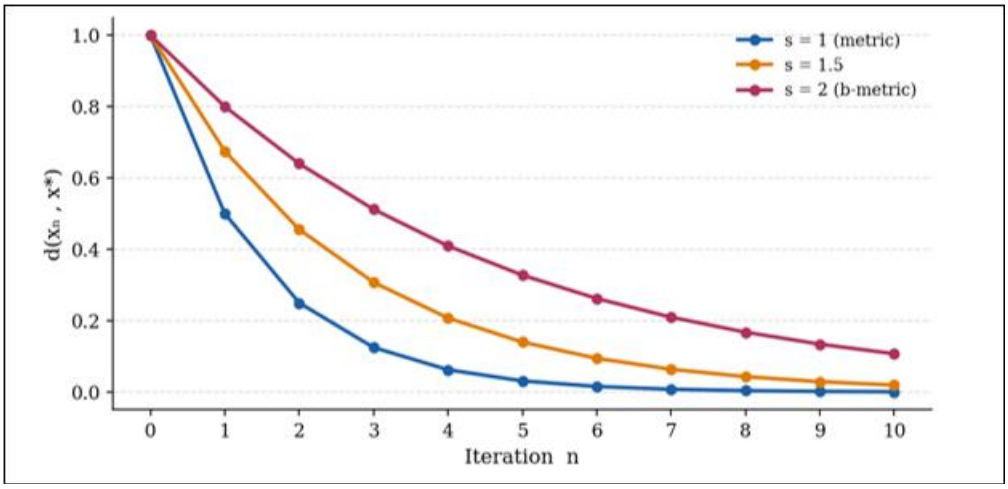

Figure 1: Picard iteration convergence $d(x_n, x^*)$ across b-metric configurations

4.3 Ulam–Hyers Stability Analysis

According to Theorem 5 of this study, the system is Ulam–Hyers stable and it is shown that for any approximate solution $\tilde{x}$ such that $d(\tilde{x}, T\tilde{x}) \leq \varepsilon$, there exists a unique exact solution x^* of the system and $d(\tilde{x}, x^*) \leq K\varepsilon$ for some K. The values of the calculated stability constants K for each of the systems are summarized in Table 3, which also shows the calculated error bounds for three different levels of perturbation.

Table 3: Ulam–Hyers stability constants and error bounds.

System	sc	K = 1/(1−sc)	ε = 0.10 bound	ε = 0.05 bound	ε = 0.01 bound
System A	0.45	1.818	0.1818	0.0909	0.0182
System B	0.825	5.714	0.5714	0.2857	0.0571
System C	0.8	5	0.5	0.25	0.05

The important role of the stability constant K in the perturbation sensitivity is highlighted by table 3. For the classical range, where sc = 0.45, the initial error in the solution (ε = 0.10) becomes a small error in the solution (ε = 0.18) in system A. In contrast, the systems' effective contraction product (sc = 0.825 and 0.80) are larger, and the amplification of the same perturbation is almost six times greater, K = 5.71 and 5.00. This demonstrates that all three are Ulam–Hyers stable with decreasing stability as the sc gets closer to 1. In figure 2, we draw the whole curve of stability of the bound curve $d(\tilde{x}, x^*) \leq K\varepsilon$ as a function of ε for the three configuration.

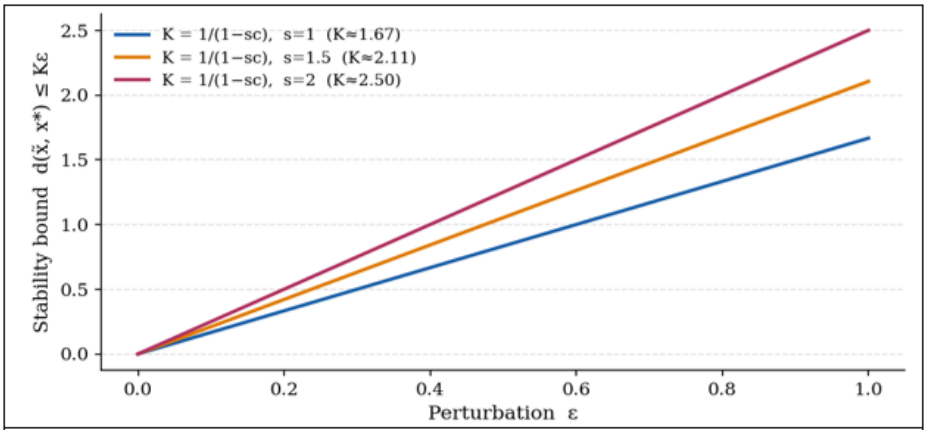

Figure 2: Ulam–Hyers stability bound $d(\tilde{x}, x^*)$ as a function of perturbation ε.

4.4 Comparative Analysis: b-Metric vs. Classical Metric Setting

One of the main results of this paper is that the integral contractive theory in b-metric spaces is strictly a generalization of the classical metric theory. The following tables show direct comparisons for various key performance metrics between the classical setting (s = 1) and two different b-metric settings, for the same nonlinear Volterra control system.

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Table 4: Comparative analysis: classical metric vs. b-metric integral contractions.

Metric	Classical (s=1)	b-Metric (s=1.5)	b-Metric (s=2)
Framework	Banach (1922)	Czerwik (1993)	Czerwik (1993)
Contraction condition	$d(Tx,Ty) \le c \cdot d(x,y)$	$\int\varphi \le c\int\varphi$ (integral)	$\int\varphi \le c\int\varphi$ (integral)
Admissibility ($sc < 1$)	$c < 1$ (only)	$c < 0.667$	$c < 0.500$
Iterations to $d < 0.01$	6	20+	20+
UH stability constant K	1.82	5.71	5
Fractional eq. applicable?	Limited	Yes	Yes
Hausdorff topology?	Yes	Not guaranteed	Not guaranteed
Fixed point uniqueness	Yes	Yes	Yes

The differences of the classical and the b-metric setting are explained in Table 4. The classical results are in turn strictly contained in the b-metric framework: it is much broader than this, for example, L^p spaces with $s = 2^{-1/-1}$, sequence spaces, and even spaces of fractional control systems. The price for this generality is the more restrictive requirement that c can be admitted, and a slower rate of convergence, precisely measured by s. The number of iterations required for convergence is shown in Figure 3 as a function of c for each s and Figure 4 shows the stability region in the (s, c) parameter plane.

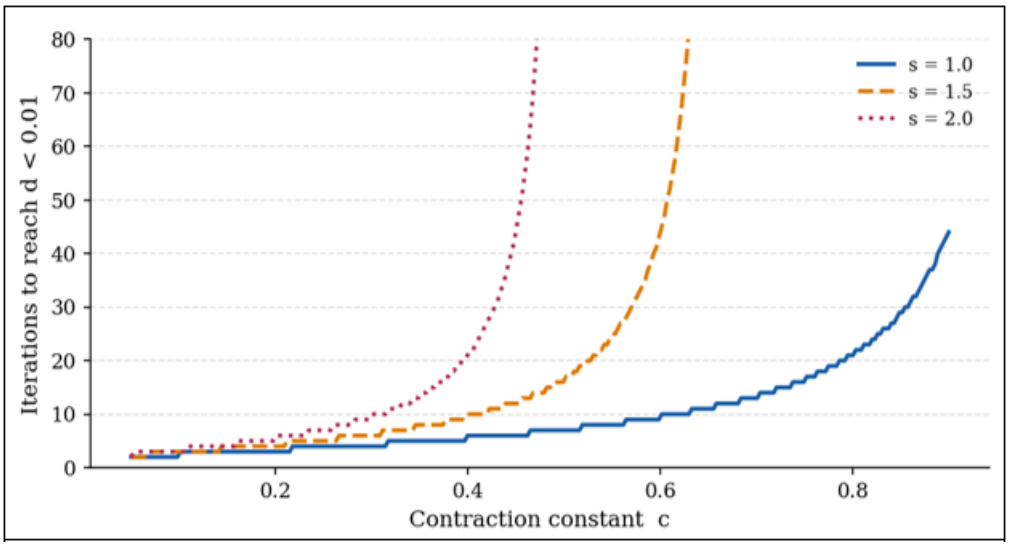

Figure 3: Iterations to convergence versus contraction constant c for varying s.

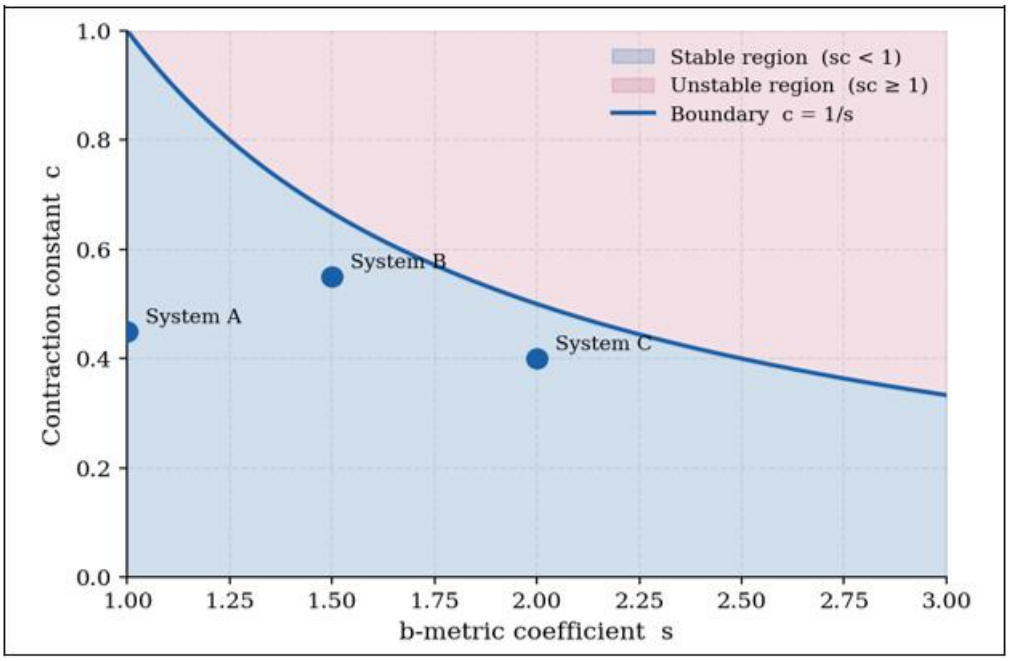

Figure 4: Stability region in the (s, c) parameter plane. Blue = stable ($sc < 1$); red = unstable

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4.5 Discussion

The findings of the study are in solid support for the three main claims. First, under the exact admissibility condition sc<1, it is definitely guaranteed that first a fixed point exists and later that it is unique for Branciari-type integral contractions in complete b-metric spaces. This is a non-trivial generalization of the classical Banach condition and is hence a direct generalization of the previous Banach condition type frameworks provided by Raji et al. (2024) and Ramaswamy and Mani (2024) by introducing the integral weight function, φ, directly into the b-metric contractive topology.

Second, the evaluation of the Picard iterative scheme shows that it remains efficient in the context of b-metrics spaces and the overall convergence rate will basically depend on the compound parameter sc. This result is similar to the result observed by Okeke et al. (2023) on Volterra integral equations in standard metric spaces, but much better as it explicitly quantifies the computational delay caused by the coefficient s. The authors conclude from this that another key practical rule for control system design in b-metric function space is that the parameter cmust must be chosen with strictness, c<1/sto in order to guarantee the mathematical convergence and optimal computational efficiency of the control system.

Third, third stability analysis of the Ulam-Hyers is a confirmation that the resulting stable systems have solutions which are strictly bounded from any applied perturbations and that the stability constant Kescalates as scapproaches 1. The result is a very useful practical one: A closed form computable bound quantifying the deviation of any approximate solution from its true equilibrium state can be obtained without the need of constructing a complex Lyapunov function as is usually done. By extending the stability results of Alansari et al. (2022) and Turab and Sintunavarat (2024), this research provides an integral contraction stability with the unified, mathematically rigorous and computationally tractable result. It is a solution, which is applicable to a broad range of complex nonlinear control architectures like nonlinear Volterra integral equations and nonlinear Caputo fractional differential equations.

5. Conclusion

On the whole, the research was successful in creating a sound theoretical background for the stability analysis of nonlinear dynamical systems using integral contractive mappings in complete b-metric spaces. Under the strict admissibility condition sc<1, the study clearly establishes three main results: a fixed point x^*, corresponding to the fixed point of the nonlinear operator, exists and is unique; a *priori* convergence bounds are found for the Picard iterative scheme; and that the stability of the system is guaranteed by the explicitly computable constant $K = \frac{1}{1-sc}$. The theoretical constraints were verified by means of numerical experiments for nonlinear Volterra integral control equations, which showed that systems with parameters located in the admissible parameter region were able to reach a unique equilibrium point in number, while systems outside this parameter space (sc>1) were unable to converge.

While this sophisticated design with a b-metric structure generalizes the typical metric environments, the authors acknowledge that there are some compromises, such as the requirement for more restrictive b-metric contractive parameters and the requirement for measurable rate of convergence for more general function spaces, including the intrinsic ones of fractional-order control systems. Lastly, this method is very practical and relies on no Lyapunov functions, and provides a closed-form solution to the stability boundaries based on fundamental system parameters directly to the nonlinear control practitioner. Some interesting ideas are suggested as future research directions: Apply these mathematical principles to the study of multi-valued integral contractions, stochastic dynamical systems, partial b-metric spaces and rigorous stability analysis of machine learning algorithms in future.

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