International Journal of Scientific Engineering and Research (IJSER)
ISSN (Online): 2347-3878
SJIF (2025): 8.036

# A Review on Classical and Generalized Fixed Point Theorems and their Applications to Fractional Differential Equations

**P. Thirupathi Rao[1], Adilakshmi Gorantla[2]**

[1] Research Scholar, Department of Mathematics, Dr. RVR NRI Institute of Technology (Deemed to be University), Pothavarappadu (V), Agiripalli (M) -521212, Vijayawada Rural, Andhra Pradesh, India
Email: *ptr8085[at]gmail.com*

[2]Professor, Department of Mathematics, Dr. RVR NRI Institute of Technology (Deemed to be University), Pothavarappadu (V), Agiripalli (M) -521212, Vijayawada Rural, Andhra Pradesh, India
*Corresponding Author* Email: *ssm.adilakshmi[at]gmail.com*

**Abstract:** ***Fixed point theory seems pretty central in nonlinear analysis these days. It shows up in differential equations, optimization, economics and a few other spots too. Banach started the main idea with his contraction principle in 1922, and after that the whole area grew fast. People like Kannan, Chatterjea, Reich and others each added their own contractive conditions, which let the theory cover more mappings than before. It feels like those changes made fixed points useful in places it did not reach earlier. Some of the newer versions use rational expressions or theta type mappings, and those have turned up in fractional differential equations along with market models. I think the review tries to trace how the ideas changed over time and lines up the different assumptions. It also looks at recent work and how it connects to fractional calculus. The comparisons between all the conditions can get a bit messy though, and not every detail ends up completely settled.***

**Keywords:** Fixed point theorem, Banach contraction, Kannan contraction, Reich contraction, Fractional differential equations, ɵ-contractions

## 1. Introduction

Fixed point theory is important in analysis these days. It helps show solutions exist and stay unique for nonlinear equations and similar things.

The Banach Contraction Principle from 1922 seems to be the main starting point here. It sets up conditions for a unique fixed point in complete metric spaces. This covers integral equations and differential ones along with optimization problems too. I think that part gets a bit messy when you try to see all the connections at once. Some people see it one way others do not [1].

Since Banach's pioneering work, researchers have proposed numerous generalizations that weaken the contraction condition while preserving fixed point existence. Notable developments include the contractions introduced by Kannan [2], Reich [4], Sehgal [5], Edelstein [6], and Chatterjea [3]. These extensions significantly increased the applicability of fixed point methods in nonlinear analysis.

Recent research has focused on generalized contractions such as $\theta$-contractions and rational contractions. These modern approaches have demonstrated effectiveness in solving fractional differential equations and mathematical models arising in economics and engineering [8]. Fractional calculus itself has become an important field due to its ability to model memory and hereditary properties of various physical systems [9, 10].

This review presents the historical evolution of these contraction principles and discusses their applications.

## 2. Banach Contraction

Banach introduced the contraction mapping theorem in complete metric spaces, establishing one of the most important results in functional analysis [1].

Let $(X,d)$ be a complete metric space. If $d(Tx,Ty) \le kd(x,y)$, $0 < k < 1$, then $T$ possesses a unique fixed point.

The Banach theorem has applications in
- Differential equations
- Integral equations
- Functional equations
- Numerical analysis
- Dynamical systems

Its simplicity has made it the foundation of modern fixed point theory.

## 3. Generalizations of banach's principle

### 3.1 Edelstein Contraction

Edelstein introduced a weaker notion of contraction for compact metric spaces by requiring

$$d(Tx,Ty) < d(x,y),$$

for every distinct pair of points [6].

Unlike Banach's theorem, no global contraction constant is required.

### 3.2 Kannan Contraction

Kannan proposed a contraction independent of the distance between the points themselves [2]. A mapping satisfies

$$d(Tx,Ty) \le \alpha[d(x,Tx) + d(y,Ty)],$$

Where

$$0 \le \alpha < \frac{1}{2}.$$

This condition is independent of Banach contractions, making the theorem applicable to broader classes of nonlinear operators.

### 3.3 Reich Contraction

Reich further generalized Kannan's theorem by combining several distance terms [4]. His condition includes

$$d(x,y)\ ,\ d(x,Tx)\ ,\ d(y,Ty)$$

allowing greater flexibility in proving fixed point results.

### 3.4 Sehgal's Contractive Iterate

Sehgal considered mappings whose suitable iterates become contractive instead of requiring the mapping itself to satisfy a contraction [5].

This approach considerably enlarges the class of admissible mappings.

### 3.5 Chatterjea Contraction

Chatterjea introduced another important contraction of the form [3]

$d(Tx,Ty) \le \alpha[d(x,Ty) + d(y,Tx)]$, where $0 \le \alpha < \frac{1}{2}.$

This theorem represents another independent generalization of Banach's contraction principle.

## 4. Recent Developments

Recent investigations continue to generalize classical contractions.

Bataihah [7] proposed a displacement functional approach for strictly contractive mappings and Kannan mappings. The work simplifies the proofs of several fixed point results while extending their applicability.

Hammad and Albeladi [8] introduced rational $\theta$-Reich and $\theta$-Sehgal contractions. Their work established new existence and uniqueness results together with applications to

- Fractional systems,
- Market equilibrium,
- Economic growth models.

These developments demonstrate that generalized contractions remain an active area of mathematical research.

## 5. Applications to Fractional Differential Equations

Fractional differential equations model systems possessing memory and hereditary characteristics.

Podlubny [9] provided one of the earliest comprehensive treatments of fractional differential equations, covering

- Caputo derivatives,
- Riemann–Liouville derivatives,
- Existence and uniqueness theory.

Kilbas, Srivastava, and Trujillo [10] further developed the mathematical foundations and applications of fractional calculus.

Many existence results for fractional differential equations rely on fixed point techniques because nonlinear fractional operators naturally satisfy generalized contraction conditions.

| Theorem | Year | Main Condition | Improvement |
|---|---|---|---|
| Banach | 1922 | Lipschitz contraction | Classical result |
| Edelstein | 1962 | Strict contraction | No contraction constant |
| Kannan | 1968 | Distance to images | Independent of Banach |
| Sehgal | 1969 | Contractive iterate | Larger mapping class |
| Reich | 1971 | Mixed distance condition | Generalized contraction |
| Chatterjea | 1972 | Cross-distance condition | Alternative contraction |
| Bataihah | 2026 | Displacement functional | Modern framework |
| Hammad & Albeladi | 2026 | Rational θ-contractions | Fractional applications |

## 6. Future Research Directions

Future research may focus on

- Ffuzzy metric spaces,
- Ppartial metric spaces,
- b-metric spaces,
- Graph metric spaces,
- Probabilistic metric spaces,
- Fractional integro-differential equations,
- Machine learning optimization,
- Economic equilibrium models.

Generalized contraction principles are expected to remain important tools in nonlinear analysis.

## 7. Conclusion

The evolution from Banach's contraction principle to modern $\theta$-contractions illustrates the remarkable development of fixed- point theory during the past century. Each generalization has weakened the contraction assumptions while maintaining fixed point existence and uniqueness. Recent studies have connected these abstract mathematical concepts with practical applications in fractional differential equations and economic modeling. Consequently, generalized fixed point theory continues to be an active and influential research area.

## References


[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, *Fundamenta Mathematicae*, 3 (1922), 133–181.

[2] R. Kannan, Some results on fixed points, *Bulletin of the Calcutta Mathematical Society*, 60 (1968), 71–76.

[3] S. K. Chatterjea, Fixed point theorems, *Comptes Rendus de l'Academie Bulgare des Sciences*, 25 (1972), 727–730.

[4] S. Reich, Kannan's fixed point theorem, *Bollettino dell'Unione Matematica Italiana*, 4 (1971), 1–11.

[5] V. M. Sehgal, A fixed point theorem for mappings with a contractive iterate, *Proceedings of the American Mathematical Society*, 23 (1969), 631–634.

[6] M. Edelstein, On fixed and periodic points under contractive mappings, *Journal of the London Mathematical Society*, 37 (1962), 74–79.

[7] A. Bataihah, A displacement functional approach to fixed points of strictly contractive and Kannan mappings, *International Journal of Analysis and Applications*, 24 (2026), 174.

[8] H. A. Hammad and G. Albeladi, Rational $\theta$-Reich and $\theta$-Sehgal contractions and relevant fixed points with applications to fractional systems, market equilibrium, and economic growth models, *Journal of Mathematics*, 2026.

[9] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.


## Author Profile

**Dr. P. Thirupathi Rao** is an Associate Professor of Mathematics with over 33 years of teaching experience in higher education. He obtained his M.Sc. in Mathematics from Kakatiya University and M.Phil. in Mathematics from Madurai Kamaraj University. He is the author of *A Text Book of Probability and Statistics* published by Laxmi Publications, New Delhi.

## Corresponding Author Profile

**Dr. Gorantla Adilakshmi** is a Professor of Mathematics at Dr. RVR NRI Institute of Technology(Deemed to be University), Andhra Pradesh, India. She obtained her Ph.D. in Mathematics from Koneru Lakshmaiah Education Foundation (KLEF) in 2021, specializing in Fixed Point Theory and its applications in advanced metric spaces. With over 16 years of teaching and research experience, her research interests include Fixed Point Theory, Functional Analysis, Graph Theory, Recommender Systems, Artificial Intelligence, and Quantum Computing. She has published 15 Scopus-indexed research papers in reputed international journals and conferences. Dr. Adilakshmi serves as SPOC for the Quantum Innovation Centre and PMKVY projects and has received several recognitions, including the Dr. Sarvepalli Radhakrishnan Best Teacher Award and the Young Scientist Award in the Defence Science Conclave -2026.