Quartic Rander's Change of Finsler Metric

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Abstract: The purpose of the present paper is to study the $S_4$-likeness of Quartic Rander's change of a Finsler space and the relation between $V$-curvature tensor of Quartic and its Quartic Rander's changed Finsler space.

Keywords: $V$-curvature tensor, $S_3$-likeness, $S_4$-likeness, Quartic Finsler space

1. Introduction

Let $M^n$ be an n-dimensional differentiable manifold and $F^n$ be a Finsler space equipped with a fundamental function $\alpha(x, y)$, $(y_i = y^i)$ of $M^n$. If a differential 1-form $\beta(x, y) = b_i(x) y^i$ is given on $M^n$, then M. Matsumoto [4] introduced another Finsler space whose fundamental function is given by

$$L(x, y) = \alpha(x, y) + \beta(x, y) \quad (1.1)$$

This change of Finsler metric has been called $\beta$-change [11], [12]. If $\alpha(x, y)$ is a Riemannian metric, then the Finsler space with a metric $L = \alpha + \beta$ where $\mu = \{a_{ij}(y)y^iy^j\}^{1/2}$ is a Riemannian metric. This metric was introduced by G. Rander's [10]. In papers [1], [2], [3], [5] and [7] Randers spaces have been studied from a geometrical view point and various theorems were obtained. In 1978 S. Numata [9] introduced another $\beta$-change of Finsler metric given by $L = \mu + \beta$ where $\mu = \{a_{ij}(y)y^iy^j\}^{1/2}$ is a Minkowski metric and $\beta$ is as above. This metric is of the similar form of Rander's one, but there are different tensor properties, because the Riemannian space with the metric $\alpha$ is characterized by $C_{ij}^k = 0$ and on the other hand the locally Minkowski space with metric $\mu$ by $R_{ijk}=0$, $C_{ijk}=0$.

In 1978 M. Matsumoto and S. Numata [8] introduced the so called cubic metric on a differential manifold with the local coordinate $x^i$ defined by

$$L = \{a_{ijk}(x)y^iy^jy^k\}^{1/3} \quad (2.1)$$

where $a_{ijk}(x)$ are component of a symmetric tensor field of $(0, 3)$ type depending on the position $x$ alone and has been called a cubic Finsler space. This cubic metric is of the similar form to the Riemannian metric $\alpha$, which is characterized by $\delta\delta\delta\alpha = 0$, whereas cubic metric L is characterized by $\delta\delta\delta\delta\delta\alpha = 0$, $\delta\delta\delta\alpha = 0$.

In the present paper we shall introduced a Finsler space with a metric

$$L(x, y) = L(x, y) + \beta(x, y) \quad (1.2)$$

This metric is of the similar form to the Rander’s one in the sense that the Riemannian metric is replaced with the Quartic metric, that is, why we will call the change (1.2) as Quartic Randers change of Finsler metric. The relation between $v$-curvature tensor of Quartic Finsler space and its Quartic Rander’s changed Finsler space has been obtained.

2. The Fundamental Tensors of $F^n$

We consider an n-dimensional Finsler space $F^n$ with a metric $L(x, y)$ given by $s$

$$(2.1) \quad \ell(x, y) = L(x, y) + b(x)y^i$$

where

$$(2.2) \quad L = a_{ijk}(x)y^iy^jy^k$$

By putting

$$(2.3) \quad a_{ijk} = \frac{a_{ijk}y^iy^jy^k}{L^3}, \quad a_{ij} = \frac{a_{ijk}y^iy^k}{L^2}$$

We obtained the normalized element of support $L = \delta L$ and the angular tensor metric $h_{ij} = \frac{\delta h_{ij}}{L}$ as

$$(2.4) \quad h_{ij} = a_i \delta b_j$$

$$(2.5) \quad b_i = \frac{\delta b_i}{L}$$

where $h_{ij}$ is the angular metric tensor of Quartic Finsler space with metric $\ell$ given by

$$(2.6) \quad h_{ij} = 3(a_{ij} - a_{ai}a_{aj})$$

The fundamental metric tensor $g_{ij} = \delta g_{ij}(\frac{\ell^2}{2}) = \delta g_{ij} + \delta L$ of Finsler space $F^n$ are obtained from equations (2.4), (2.5) and (2.6) which is given by

$$(2.7) \quad \delta g_{ij} = 3(1 - 3t)a_{ij} + (a_{ib} + a_{bi}) + b_i + b_j$$

where $t$ is defined by

$$(2.8) \quad \delta g_{ij} = \delta \delta (\frac{\ell^2}{2}) = \delta g_{ij} + \delta L$$

It is easy to show that

$$\delta g_{ij} = \delta \delta (\frac{\ell^2}{2}) = \delta g_{ij} + \delta L$$

Therefore from (2.7), it follows (h) hv-torsion tension tensor $\delta C_{ijk} = \delta \delta (\frac{\ell^2}{2})$ of the Cartan’s connection $\delta g_{ij}$ are given by

$$(2.9) \quad \delta C_{ijk} = 6(1 - 3t)(a_{ik}a_{ij} + a_{ij}a_{ik}) + 3(a_{ib} + a_{bi}) + a_i b_j(a_{ij} - a_{ai}a_{aj}) - 3(a_{ia}b_j - a_{ia}b_j) + a_i b_j + a_j b_i + 3(\delta - 3t)a_i a_j$$

In view of equation (2.6) the equation (2.8) may be written as
(2.9) \( C_{ijk} = \tau C_{ijk} + (h_{ijm} + h_{ikm} + h_{ikm}) / 2L \)

where \( m = b_{ij} - \frac{E}{J} a_{ij} \) and \( C_{ijk} \) is the \((h)\) hv-torsion tensor of the Cartan’s connection CT of Quartic Finsler metric L given by

(2.10) \( LC_{ijk} = \{a_{ijk} + (a_{ijk} + a_{jik} + a_{ikj}) + 2a_{iak}a_{kij} \}

Let us suppose that the intrinsic metric tensor \( a_{ij}(x,y) \) of the Quartic metric L has non-vanishing determinant. Then the reciprocal metric tensor \( g^{ij} \) of \( F^n \) is obtain from equation (2.7) which is given by

(2.11) \( \frac{1}{g_{ij}} = a^{ij} \left( \frac{E_0^2 - 1}{E_0^2} \right) \frac{1}{L} \frac{1}{a^{ij} \left( \frac{E_0^2 - 1}{E_0^2} \right)} \)

where \( E_0 = \frac{b}{b - \frac{E}{J}} \), \( a^{ij} = a^{ij}a_{ij} \), \( b^2 = b^2 - q^2 \), \( q = aibi = \frac{aibi}{aibi} \)

To find the v-curvature tensor of \( F^n \), first we find (h) hv-torsion tensor \( C_{ijk} \), where \( a_{ijk} = a_{ijk}a_{ijk} = a_{ijk}a_{ijk} \), \( C_{ijk} = 0 \), \( h_{ijm} = 0 \), \( m_{ij} = 0 \), \( h_{imj} = 3m_{ij}, m_{ij} = (b^2 - q^2) \).

From (3.1) and (3.3) we have the following identities

(3.1) \( a_{ijk} = a_{ijk}a_{ijk} = a_{ijk}a_{ijk} \), \( C_{ijk} = 0 \), \( h_{ijm} = 0 \), \( m_{ij} = 0 \), \( h_{imj} = 3m_{ij}, m_{ij} = (b^2 - q^2) \).

Now we shall find the v-curvature tensor \( S_{ijk} \) with respect to Carton’s connection CT is of the form (3.6).

Thus (3.6) may be written as

(3.8) \( S_{ijk} = \frac{1}{3} (h_{ijk}) \left( h_{imj} + h_{ijm} + h_{ikm} \right) \)

It is well known [6] that the v−curvature tensor of any three dimensional Finsler space is of the form

(3.9) \( L^2S_{ijk} = S(h_{ijk}) \left( h_{imj} + h_{ijm} + h_{ikm} \right) \)

Owing to this fact M. Matsumoto [6] defined the \( S_3 \)-like Finsler space \( F^n \) with \( v−curvature tensor of the form \( S_4 \)-like Finsler space. When \( v\)-curvature tensor of any four dimensional Finsler space may be written as [6]

(3.10) \( L^2S_{ijk} = \left( \begin{array}{c} h_{ijk} \left( h_{imj} \right) + \left( h_{ijm} \right) + \left( h_{ikm} \right) \end{array} \right) \)

where \( K_{ij} \) is a (0, 2) type symmetric Finsler tensor field which is such that \( K_{ij} \left( x, y \right) : 0 \). A Finsler space \( F^n \) with \( v−curvature tensor of the form \( (3.10) \).

From (3.8), (3.9), (3.10) and (2.5), we have the following theorems.

Theorem 3.1: The Quartic Rander’s change of \( S_3 \)-like or \( S_4 \)-like Finsler space is \( S_4 \)-like Finsler space.

Theorem 3.2: If \( v−curvature tensor of Quartic Rander’s changed Finsler space \( F^n \) vanishes identically, then the Quartic Finsler space \( F^n \) is \( S_4 \)-like. If \( v−curvature tensor of Quartic Finsler space \( F^n \) vanishes then equation (3.8) reduces to

(3.11) \( S_{ijk} = h_{ijk} + h_{ijk} + h_{ijk} + h_{ijk} \)

By virtue of (3.11) and (2.11) and the Ricci tensor \( S_{ij} \),

(3.12) \( S_{ij} = h_{ijk} + h_{ijk} + h_{ijk} + h_{ijk} \)

where \( H_1 = m_{ij} + \left( n - 3 \right) \left( \frac{a^2 - b^2}{4L^2} \right) \), \( H_2 = \left( a - b \right) \), \( H_3 = \left( a - b \right) \).

From (3.12), we have the following:

Theorem 3.3: If the \( v−curvature tensor of Quartic Finsler space vanishes then there exist scalar \( H_1 \) and \( H_2 \) in Quartic Rander’s changed Finsler space \( F^n \) such that matrix \( |S_{ijk} + H_{ijk} + H_{ijk} + H_{ijk}| \) is of rank two.

References

