International Journal of Scientific Engineering and Research (IJSER)
www.ijser.in
ISSN (Online): 2347-3878
Volume 1 Issue 4, December 2013

Quartic Rander's Change of Finsler Metric

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Abstract: ***The purpose of the present paper is to study the S_4-likeness of Quartic Rander's change of a Finsler space and the relation between V –curvature tensor of Quartic and its Quartic Rander's changed Finsler space.***

Keywords: V −curvature tensor, S_3−likeness, S_4−likeness, Quartic Finsler space

1. Introduction

Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function α (x; y), ($y^i = \dot{x}^i$) of M^n. If a differential 1-form β (x; y) = $b_i(x)y^i$ is given on M^n, then M. Matsumoto [4] introduced another Finsler space whose fundamental function is given by

(1.1) $L(x,y) = \alpha(x,y) + \beta(x,y)$

This change of Finsler metric has been called β−change [11], [12]. If α(x,y) is a Riemannian metric, then the Finsler space with a metric L = α + β where $\alpha = \{a_{ij}(x)y^iy^j\}^{1/2}$ is a Riemannian metric. This metric was introduced by G. Rander's [10]. In papers [1], [2], [3], [5] and [7] Randers spaces have been studied from a geometrical view point and various theorems were obtained. In 1978 S. Numata [9] introduced another β−change of Finsler metric given by L = μ + β where $\mu = \{a_{ij}(y)y^iy^j\}^{1/2}$ is a Minkowski metric and β is as above. This metric is of the similar form of Rander's one, but there are different tensor properties, because the Riemannian space with the metric α is characterized by $C^i_{jk}= 0$ and on the other hand the locally Minkowski space with metric μ by $R_{hijk}= 0$, $C_{hij|k}= 0$.

In 1978 M. Matsumoto and S. Numata [8] introduced the so called cubic metric on a differential manifold with the local coordinate x^i defined by

$$L = \{a_{ijk}(x)y^iy^jy^k\}^{1/3} \quad (y^i= \dot{x}^i)$$

where $a_{ijk}(x)$ are component of a symmetric tensor field of (0,3) type depending on the position x alone and has been called a cubic Finsler space. This cubic metric is of the similar form to the Riemannian metric α, which is characterized by $\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\alpha^2= 0$, whereas cubic metric L is characterized by $\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k \dot{\partial}_pL^3= 0$.

In the present paper we shall introduced a Finsler space with a metric

(1.2) $\bar{L}(x,y) = L(x,y) + \beta(x,y)$

This metric is of the similar form to the Rander's one in the sense that the Riemannian metric is replaced with the Quartic metric, that is, why we will call the change (1.2) as Quartic Randers change of Finsler metric. The relation between v-curvature tensor of Quartic Finsler space and its Quartic Rander's changed Finsler space has been obtained.

2. The Fundamental Tensors of F^n

We consider an n-dimensional Finsler space F^n with a metic $\bar{L}$ (x,y) given by s

(2.1) $\bar{L}(x,y) = L(x,y) + b_i(x)y^i$

where

(2.2) $L^4= a_{ijkp}(x)y^iy^j\mathbf{y}^ky^p$

By putting

(2.3) $a_{ijk} = \frac{a_{ijkh}\ y^r}{L}$, $a_{ij} = \frac{a_{ijkr}\ y^K y^r}{L^2}$,
$a_i = \frac{a_{ijkr}\ y^j\ y^k y^r}{L^3}$

We obtained the normalized element of support $\bar{l}_i=\dot{\partial}_i\bar{L}$ and the angular metric tensor

$\bar{h}_{ij} = \bar{L}\,\dot{\partial}_i\dot{\partial}_j\,\bar{L}$ as

(2.4) $\bar{l}_I = a_i+ b_i$

(2.5) $\frac{h_{ij}}{L} = \frac{\bar{h}_{ij}}{\bar{L}}$

where h_{ij} is the angular metric tensor of Quartic Finsler space with metric L given by

(2.6) $h_{ij}= 3(a_{ij}- a_ia_j)$.

The fundamental metric tensor $\bar{g}_{ij} =\dot{\partial}_i\dot{\partial}_j(\frac{\bar{L}^2}{2}) = \bar{h}_{ij} +\bar{l}_i\bar{l}_j$ of Finsler space F^n are obtained from equations (2.4), (2.5) and (2.6) which is given by

(2.7) $\bar{g}_{ij} = 3\tau a_{ij}+ (1 - 3\tau)a_ia_j+ (a_ib_j+ a_jb_i) + b_ib_j$ where $\tau = \frac{\bar{L}}{L}$

It is easy to show that

$\dot{\partial}_i\tau = \frac{\{(1-\tau)a_i+ b_i\}}{L}$, $\dot{\partial}_ja_i = \frac{3(a_{ij}-a_ia_j)}{L}$,
$\dot{\partial}_k a_{ij} = \frac{2(a_{ijk}-a_{ij}a_k)}{L}$

Therefore from (2.7), it follows (h) hv-torsion tension tensor $\bar{C}_{ijk} =\dot{\partial}_k \frac{\bar{g}_{ij}}{2}$ of the Cartan's connection CΓ are given by

(2.8) $2L\bar{C}_{ijk}= 6\tau a_{ijk}+3(1-3\tau)(a_{jk}a_i+a_{ij}a_k+a_{ki}a_j)+3(a_{ij}b_k+a_{jk}b_i+a_{ki}b_j)\ -3(a_ia_jb_k+ a_ia_kb_j+ a_ja_kb_i) + 3(7\tau - 3)a_ia_ja_k$

In view of equation (2.6) the equation (2.8) may be written as

(2.9) $\bar{C}_{ijk} = \tau\, C_{ijk} + (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)\,/2L$

where $m_i = b_i - \frac{\beta}{L}\, a_i$ and C_{ijk} is the (h) hv-torsion tensor of the Cartan's connection

$C\Gamma$ of Quartic Finsler metric L given by

(2.10) $LC_{ijk} = 3\{a_{ijk} - (a_{ij}a_k + a_{jk}a_i + a_{ki}a_j) + 2a_ia_ja_k\}$

Let us suppose that the intrinsic metric tensor $a_{ij}(x,y)$ of the Quartic metric L has non-vanishing determinant. Then the inverse matrix (a^{ij}) of (a_{ij}) exists. Therefore the reciprocal metric tensor $g^{ij}$of F^n is obtain from equation (2.7) which is given by

(2.11) $\bar{g}_{ij} = \frac{1}{3\tau} a^{ij} + \frac{(b^2+3\tau-1)}{3\tau(1+q)^2} a^i a^j - \frac{(a^i b^j + a^j b^i)}{3\tau(1+q)}$

where $a^i = a^{ij}a_j$, $b^i = a^{ij}b_j$, $b^2 = b^ib_i$, $q = a^ib_i = a_ib^i = \beta / L$

3. The v-Curvature Tensor of F^n

From (2.6), (2.10) and definition of m_i and a^i, we get the following identities

(3.1) $a_ia^i = 1$, $a_{ijk}a^i = a^{jk}$, $C_{ijk}a^i = 0$, $h_{ij}a^i = 0$

$m_ia^i = 0$, $h_{ij}b^j = 3m_i$, $m_ib^i = (b^2 - q^2)$

To find the v-curvature tensor of Fn, first we find (h) hv-torsion tensor

$\bar{C}^i_{jk} = \bar{g}^{ir}\, \bar{C}_{jrk}$

(3.2) $\bar{C}^i_{jk} = \frac{1}{3} C^i_{jk} + \frac{1}{6L} (h^i_jm_k + h^i_km_j + h_{jk}m^i) - \frac{a^i}{L(1+q)} \{m_jm_k + \frac{1}{6}(b^2 - q^2)h_{jk}\} - \frac{1}{3(1+q)} a^iC_{jrk}b^r$

where $LC^i_{jk} = LC_{jrk}a^{ir} = 3\{a^i_{jk} - (\delta^i_ja_k + \delta^i_ka_j + a^ia_{jk}) + 2a^ia_ja_k\}$

(3.3) $h^i_j = h_{jr}a^{ir} = 3(\delta^i_j - a^ia_j)$

$m^i = m_ra^{ir} = b^i - qa^i$ and $a^i_{jk} = a^{ir}a_{jrk}$.

From (3.1) and (3.3) we have the following identities

$C_{ijr}h^r_p = C^r_{ij}h_{pr} = 3C_{ijp}$, $C_{ijr}m^r = C_{ijr}b^r$, $m_rh^r_i = 3m_i$,

$m_im^i = (b^2 - q^2)$, $h_{ir}h^r_j = 3h_{ij}$, $h_{ir}m^r = 3m_i$.

From (2.9) and (3.2) we get after applying the identities (3.4)

(3.5) $\bar{C}_{ijr}\bar{C}^r_{hk} = \frac{\tau}{3} C_{ijr}C^r_{hk} + \frac{1}{2L}(C_{ijh}m_k + C_{ijk}m_h + C_{hjk}m_i + C_{hik}m_j) + \frac{1}{6L}(C_{ijr}h_{hk} + C_{hrk}h_{ij})b^r + \frac{1}{12LL}(b^2 - q^2)h_{ij}h_{hk} + \frac{1}{4LL}(2h_{ij}m_hm_k + 2h_{hk}m_im_j + h_{jh}m_im_k + h_{jk}m_im_h + h_{ih}m_jm_k + h_{ik}m_jm_h)$

Now we shall find the v−curvature tensor $\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}^r_{hk} - \bar{C}_{ikr}\bar{C}^r_{hj}$. The tensor is obtained from (3.5) and given by

(3.6) $\bar{S}_{hijk} = \underset{(jk)}{Q}\{ \frac{\tau}{3} C_{ijr}C^r_{hk} + h_{ij}m_{hk} + h_{hk}m_{ij} \}$

$= \frac{\tau}{3} S_{hijk} + \underset{(jk)}{Q}\{ h_{ij}m_{hk} + h_{hk}m_{ij}\}$

where

(3.7) $m_{ij} = \frac{1}{6L} \{Cijrbr + \frac{(b^2-q^2)}{4L} h_{ij} + \frac{3}{2} L^{-1}m_im_j \}$ and the symbol $\underset{(jk)}{Q}\{\ldots\}$ denotes the exchange of j,k and subtraction.

Preposition 1: The v−curvature tensor $\bar{S}_{hijk}$ of $\bar{F}^n$ with respect to Carton's connection $C\Gamma$ is of the form (3.6).

Thus (3.6) may be written as

(3.8) $\bar{S}_{hijk} = \frac{\tau}{3} S_{hijk} + \underset{(jk)}{Q}\{h_{ij}m_{hk} + h_{hk}m_{ij}\}$

It is well known [6] that the v−curvature tensor of any three dimensional Finsler space is of the form

(3.9) $L^2S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij})$

Owing to this fact M. Matsumoto [6] defined the S_3-like Finsler space $F^n(n \geq 3)$ as such a Finsler space in which v−curvature tensor is of the form (3.9). The scalar S in (3.9) is a function of x alone. The v−curvature tensor of any four dimensional Finsler space may be written as [6]

(3.10) $L^2S_{hijk} = \underset{(jk)}{Q}\{h_{hj}K_{ki} + h_{ik}h_{hj}\}$

where K_{ij} is a (0, 2) type symmetric Finsler tensor field which is such that $K_{ij}y^j = 0$. A Finsler space $F^n(n \geq 4)$ is called S_4-like Finsler space [6] if its v−curvature tensor is of the form (3.10).

From (3.8), (3.9), (3.10) and (2.5), we have the following theorems.

Theorem 3.1 : The Quartic Rander's change of S_3-like or S_4-like Finsler space is S_4-like Finsler space.

Theorem 3.2 : If v−curvature tensor of Quartic Rander's changed Finsler space $\bar{F}^n$ vanishes identically, then the Quartic Finsler space $F^n$is S_4-like. If v−curvature tensor of Quartic Finsler space F^n vanishes then equation (3.8) reduces to

(3.11) $\bar{S}_{hijk} = h_{ij}m_{hk} + h_{hk}m_{ij} - h_{ik}m_{hj} - h_{hj}m_{ik}$

By virtue of (3.11) and (2.11) and the Ricci tensor $\bar{S}_{ik} = \bar{g}^{hk}\bar{S}_{hijk}$ is of the form

$\bar{S}_{ik} = (-\frac{1}{3\tau})\, [mh_{ik} + 3(n-3)m_{ik}]$

where $m = m_{ij}a^{ij}$, which in view of (3.7) may be written as

(3.12) $\bar{S}_{ik} + H_1h_{ik} + H_2C_{ikr}b^r = H_3m_im_k$

where $H_1 = \frac{m}{3\tau} + \frac{(n-3)(b^2-q^2)}{24L^2}$

$H_2 = \frac{(n-3)}{6L}$

$H_3 = -\frac{(n-3)}{4L^2}$

From (3.12), we have the following:

Theorem 3.3 : If the v−curvature tensor of Quartic Finsler space vanishes then there exist scalar H_1 and H_2 in Quartic Rander's changed Finsler space $F^n(n \geq 4)$ such that matrix $\|S_{ik} + H_1h_{ik} + H_2C_{ikr}b^r\|$ is of rank two.

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