International Journal of Scientific Engineering and Research (IJSER)
www.ijser.in
ISSN (Online): 2347-3878
Volume 2 Issue 1, January 2014

On Certain Topological Structures of Normed Space Valued Generalized Orlicz Function Space

Narayan Prasad Pahari

Central Department of Mathematics, Tribhuvan University, University Campus, Kirtipur, Kathmandu, Nepal

Abstract: ***The aim of this paper is to introduce and study a new class $\ell_\infty(S, (T, \|\,.\,\|), \Omega, u)$ of normed space valued function space using Orlicz function Ω as a generalization of well known basic bounded sequence space ℓ_∞ studied in functional analysis. Besides the investigation of linear space structure of the class $\ell_\infty(S, (T, \|\,.\,\|), \Omega, u)$, our primarily interest is to explore the conditions pertaining to the containment relation of the class $\ell_\infty(S, (T, \|\,.\,\|), \Omega, u)$ in terms of different values of u .***

Keywords: Orlicz function, Orlicz Space, Normed Space, Solid space.

1. Introduction

So far, a good number of research works have been done on various types of Orlicz sequence space and Orlicz function spaces. In papers [1], [2] , [3], [4], [5], [6], [10], [11], [12], [13],[14],[15], [16], [17], [18] the algebraic and topological properties of several sequence and function spaces using Orlicz function have been introduced and studied as the generalizations of well known sequence spaces and function spaces.

We begin with recalling some notations and basic definitions that are used in this paper.

Definition 1.1: A function $\Omega : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it is continuous, non decreasing and convex with $\Omega(0) = 0$, $\Omega(s) > 0$ for $s > 0$, and $\Omega(s) \to \infty$ as $s \to \infty$.

An Orlicz function Ω can be represented in the following integral form

$$\Omega(s) = \int_0^s q(t)\,dt$$

where q, known as the kernel of Ω, is right-differentiable for $t \ge 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non decreasing, and $q(t) \to \infty$ as $t \to \infty$ (see, [7]).

Definition 1.2: An Orlicz function Ω is said to satisfy Δ_2-condition for all values of t, if there exists a constant

$K > 0$ such that

$$\Omega(2t) \le K\,\Omega(t)\text{ , for all } t \ge 0.$$

The Δ_2-condition is equivalent to the satisfaction of the inequality

$$\Omega(Lt) \le K\,L\,\Omega(t)$$

for all values of t for which $L > 1$, (see,[7]).

Definition 1.3: Lindenstrauss and Tzafriri [8] used the idea of Orlicz function Ω to construct the sequence space

$$\ell_\Omega = \inf\left\{\bar{s} = < s_k >: \sum_{k=1}^{\infty} \Omega\left(\frac{|s_k|}{r}\right) < \infty \text{ for some } r > 0\right\}$$

of scalars $< s_k >$,which forms a Banach space with Luxemburg norm defined by

$$\|\bar{s}\|_\Omega = \inf\left\{r > 0: \sum_{k=1}^{\infty} \Omega\left(\frac{|s_k|}{r}\right) \le 1\right\}.$$

The space ℓ_Ω is called an Orlicz sequence space and is closely related to the space ℓ_p with

$$\Omega(s) = s^p, (1 \le p < \infty).$$

It has very rich topological and geometrical properties that do not occur in ordinary ℓ_p space.

Definition 1.4: A normed space $(S, \|\,.\,\|)$ is a linear space S together with the mapping $\|\,.\,\| : S \to \mathbf{R}_+$

(Called norm on S) such that for all $s, s' \in S$ and $\alpha \in \mathbf{C}$, we have

N_1: $\|s\| \ge 0$ and $\|s\| = 0$ if and only if $s = \theta$;
N_2: $\|\alpha s\| = |\alpha|\,\|s\|$; and
N_3: $\|s + s'\| \le \|s\| + \|s'\|$.

Clearly by N_1 and N_2, algebraic operations of addition and scalar multiplication in the normed space S are continuous i.e., if $< s_n >$ and $< s'_n >$ are sequences in the normed space S with $s, s' \in S$ such that $s_n \to s$, $s'_n \to s'$ in S, and $< \alpha_n >$ a sequence of scalars with $\alpha \in C$ such that $\alpha_n \to \alpha$ in $\mathbf{C}$ then

$$s_n + s'_n \to s + s' \text{ and } \alpha_n s_n \to \alpha s \text{ in S.}$$

In fact, the algebraic and topological properties of various function spaces in normed spaces have been introduced and studied in the works of Maddox [9], Tiwari and Srivastava [19] and many others. All these function spaces generalize and unify various existing basic sequence spaces studied in Functional Analysis.

Definition 1.5: Let T be a normed space and

$$V(T) = \{ \phi : S \to T\}$$

be the classes of T-valued functions. Then V(T) is called solid if $\phi \in V(T)$ and scalars $\alpha(s), s \in S$ such that $|\alpha(s)| \leq 1$, $s \in S$ implies $\alpha(s)\, \phi(s) \in V(T)$.

2. The Class of Normed Space Valued Functions

Let S be an arbitrary non empty set (not necessarily countable) and $\mathcal{F}(S)$ be the collection of all finite subsets of S directed by inclusion relation. Let (T, ‖ . ‖) be a Banach space over the field of complex number **C.** We shall write u, v for the functions on $S \to \mathbf{C} - \{0\}$ and the collection of all such functions will be denoted by

N(S, **C** - {0}).

We now introduce the following new class of Banach space T - valued functions using Orlicz function Ω:

$$\ell_\infty (S, (T, \|.\|), \Omega, u) = \{\phi : S \to T \text{ such that } \sup_{s\in S} \Omega\left(\frac{\| u(s)\, \phi(s) \|}{r}\right) < \infty, \text{ for some } r > 0\}.$$

If in the definition of $\ell_\infty (S, (T, \|.\|), \Omega, u)$ the phrase 'for some r > 0' is replaced by 'for every r > 0' then we denote this subclass by $\ell_1 (S, (T, \|.\|), \Omega, \alpha)$.

Thus

$$\ell_1 (S, (T, \|.\|), \Omega, u) = \{ \phi : S \to T : \sup_{s\in S} \Omega\left(\frac{\| u(s)\, \phi(s) \|}{r}\right) < \infty, \text{ for every } r > 0\}.$$

Further when $u : S \to \mathbf{C} - \{0\}$ is a function such that

$$u(s) = 1 \text{ for all } s,$$

then $\ell_\infty (S, (T, \|.\|), \Omega, u)$ will be denoted by $\ell_\infty (S, (T, \|.\|), \Omega)$.

3. Main Results

In this section, we study the linear space structure of the class $\ell_\infty (S, (T, \|.\|), \Omega, u)$. As far as the linear space structures of the class $\ell_\infty (S, (T, \|.\|), \Omega, u)$ over the field of complex numbers **C** are concerned, we shall take point-wise vector operations, i.e., for any $\phi, \psi \in \ell_\infty (S, (T, \|.\|), \Omega, u)$, we have

$(\phi + \psi)(s) = \phi(s) + \psi(s)$,
and
$(\alpha \phi)(s) = \alpha\, \phi(s)$, $s \in S$, $\alpha \in C$.

Moreover, we shall denote zero element of $\ell_\infty (S, (T, \|.\|), \Omega, u)$ by θ by which we mean the function $\theta : S \to T$ such that

$$\theta(s) = 0 \text{ for all } s \in S.$$

Theorem 3.1: The class $\ell_\infty (S, (T, \|.\|), \Omega, u)$ forms a solid .

Proof:

Let $\phi \in \ell_\infty (S, (T, \|.\|), \Omega, u)$ and r > 0 be associated with ϕ.Then we have

$$\text{(3.1)} \quad \sup_{s\in S} \Omega\left(\frac{\| u(s)\, \phi(s) \|}{r}\right) < \infty.$$

Now, if we take scalars $\alpha(s)$, $s \in S$ such that $|\alpha(s)| \leq 1$, then using non-decreasing property of Ω ,

we have

$$\Omega\left(\frac{\| \alpha(s)\, u(s)\, \phi(s) \|}{r}\right) \leq \Omega\left(\frac{|\alpha(s)|\, \| u(s)\, \phi(s) \|}{r}\right) \leq \Omega\left(\frac{\| u(s)\, \phi(s) \|}{r}\right)$$

and hence in view of (3.1),

$$\sup_{s\in S} \Omega\left(\frac{\| \alpha(s)\, u(s)\, \phi(s) \|}{r}\right) \leq \sup_{s\in S} \Omega\left(\frac{\| u(s)\, \phi(s) \|}{r}\right) < \infty.$$

This shows that $\alpha \phi \in \ell_\infty (S, (T, \|.\|), \Omega, u)$ and hence $\ell_\infty (S, (T, \|.\|), \Omega, u)$ forms a solid.

Theorem 3.2: The class $\ell_\infty (S, (T, \|.\|), \Omega, u)$ forms a linear space over the field of complex numbers C with respect to the point wise vector operations.

Proof:

Let us suppose that $\phi, \psi \in \ell_\infty (S, (T, \|.\|), \Omega, u)$, $r_1 > 0$ and $r_2 > 0$ associated with

ϕ and ψ respectively and $\alpha, \beta \in \mathbf{C}$. Then we have

$$\text{(3.2)} \quad \sup_{s\in S} \Omega\left(\frac{\| u(s)\, \phi(s) \|}{r_1}\right) < \infty$$

and

$$\text{(3.3)} \quad \sup_{s\in S} \Omega\left(\frac{\| u(s)\, \psi(s) \|}{r_2}\right) < \infty .$$

We now set r_3 such that

$$r_3 = \max (2|\alpha|\, r_1, 2|\beta|\, r_2) .$$

So that

$$\frac{|\alpha|}{r_3} \leq \frac{1}{2\, r_1} \text{ and } \frac{|\beta|}{r_3} \leq \frac{1}{2\, r_2}.$$

For such r ,using non decreasing and convex properties of Ω we have

$$\Omega\left(\frac{\| u(s)\, [\alpha\, \phi(s) + \beta\, \psi(s)] \|}{r_3}\right)$$

$$\leq \Omega\left(\frac{|\alpha|\,\|u(s)\,\phi(s)\|}{r_3}+\frac{|\beta|\,\|u(s)\,\psi(s)\|}{r_3}\right)$$
$$\leq \Omega\left(\frac{\|u(s)\,\phi(s)\|}{2r_1}+\frac{\|u(s)\,\psi(s)\|}{2r_2}\right)$$
$$\leq \frac{1}{2}\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right)+\frac{1}{2}\Omega\left(\frac{\|u(s)\,\psi(s)\|}{r_2}\right)$$

Hence in view of (3.2) and (3.3),we have

$$\sup_{s\in S}\Omega\left(\frac{\|u(s)\,[\alpha\,\phi(s)+\beta\,\psi(s)]\|}{r_3}\right)$$
$$\leq \frac{1}{2}\sup_{s\in S}\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right)+\frac{1}{2}\sup_{s\in S}\Omega\left(\frac{\|u(s)\,\psi(s)\|}{r_2}\right)$$
$$< \infty,$$

and shows that

$$\alpha\,\phi+\beta\psi \in \ell_\infty\,(S, (T, \|\,.\,\|), \Omega, u).$$

Hence, ℓ_∞ (S, (T, $\|\,.\,\|$), Ω , u) forms a linear space over the field of complex numbers **C**.

Theorem 3.3: If Ω satisfies the Δ_2 – condition then we have

$$\ell_\infty\,(S, (T, \|\,.\,\|), \Omega, u) = \ell_1\,(S, (T, \|\,.\,\|), \Omega, u).$$

Proof:

Obviously by definition of ℓ_1 (S, (T, $\|\,.\,\|$) , Ω , u),

$$\ell_\infty\,(S, (T, \|\,.\,\|), \Omega, u) \supseteq \ell_1\,(S, (T, \|\,.\,\|), \Omega, u).$$

Hence to prove the assertion completely, it is sufficient to show that ℓ_∞ (S, (T, $\|\,.\,\|$) , Ω , u) is a subset of

ℓ_1 (S, (T, $\|\,.\,\|$) , Ω , u).

Suppose $\phi \in \ell_\infty$ (S, (T, $\|\,.\,\|$) , Ω , u) then for some

$r_1 > 0$,

$$\sup_{s\in S}\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right) < \infty.$$

Let $r_2 > 0$. If $r_1 \leq r_2$ then clearly by the non decreasing property of Ω, we have

$$\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_2}\right) < \Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right)$$

and hence

$$\sup_{s\in S}\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_2}\right) < \sup_{s\in S}\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right)$$
$$< \infty.$$

On the other hand, if $r_2 < r_1$ then $\frac{r_1}{r_2} > 1$.

Since, Ω satisfies the Δ_2–condition, there exists a constant $K > 0$ such that

$$\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_2}\right) = \Omega\left(\frac{r_1}{r_2}\,.\,\frac{\|u(s)\,\phi(s)\|}{r_1}\right)$$
$$\leq K\,\frac{r_1}{r_2}\,\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right)$$

for each $s \in S$, which implies that

$$\sup_{s\in S}\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_2}\right) < \infty, \text{ for every } r_2 > 0.$$

This shows that

$$\ell_\infty\,(S, (T, \|\,.\,\|), \Omega, u) \subseteq \ell_1\,(S, (T, \|\,.\,\|), \Omega, u)$$

and hence

$$\ell_\infty\,(S, (T, \|\,.\,\|), \Omega, u) = \ell_1\,(S, (T, \|\,.\,\|), \Omega, u).$$

This completes the proof.

Corollary 3.4: If Ω satisfies the Δ_2 -condition, then ℓ_1 (S, (T, $\|\,.\,\|$), Ω, u) forms linear space over **C**.

Proof:

Proof follows from the immediate consequence of Theorems 3.2 and 3.3.

In the followings, we shall investigate some of the results that characterize the condition pertaining to the containment relation of the class ℓ_∞ (S, (T, $\|\,.\,\|$), Ω , u) in terms of different values of u and thereby derive the condition of their equality.

Theorem 3.5: For any u, v $\in$ N (S, **C** - {0}),

$$\ell_\infty\,(S, (T, \|\,.\,\|), \Omega, u) \subseteq \ell_\infty\,(S, (T, \|\,.\,\|), \Omega, v)$$

if $\liminf_s \left|\frac{u(s)}{v(s)}\right| > 0$.

Proof:

Assume that

$$\liminf_s \left|\frac{u(s)}{v(s)}\right| > 0.$$

Then there exists a positive constant m such that

$$m\,|\,v(s)| < |\,u(s)|,$$

for all but finitely many $s \in S$.

Let $\phi \in \ell_\infty$ (S, (T, $\|\,.\,\|$) , Ω, u) and $r_1 > 0$ be associated with ϕ, so that

$$\sup_{s\in S}\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right) < \infty. \tag{3.4}$$

Let us choose r such that $r_1 < m\,r$. For such r ,using non decreasing property of Ω, we have

$$\Omega\left(\frac{\|v(s)\,\phi(s)\|}{r}\right) = \Omega\left(\frac{|v(s)|\,\|\phi(s)\|}{r}\right)$$
$$\leq \Omega\left(\frac{|u(s)|\,\|\phi(s)\|}{mr}\right)$$
$$\leq \Omega\left(\frac{\|u(s)\,\phi(s)\|}{r_1}\right)$$

International Journal of Scientific Engineering and Research (IJSER)
www.ijser.in
ISSN (Online): 2347-3878
Volume 2 Issue 1, January 2014

and hence in view of (3.4),we have

$$\sup_{s\in S} \Omega\left(\frac{\| v(s)\,\phi(s)\|}{r}\right) \le \sup_{s\in S} \Omega\left(\frac{\| u(s)\,\phi(s)\|}{r_1}\right) < \infty.$$

This shows that $\phi \in \ell_\infty(S, (T, \|\,.\,\|), \Omega, v)$ and hence

$$\ell_\infty(S, (T, \|\,.\,\|), \Omega, u) \subseteq \ell_\infty(S, (T, \|\,.\,\|), \Omega, v).$$

This completes the proof.

Theorem 3.6: For any $u, v \in N(S, \mathbf{C} - \{0\})$,

$$\ell_\infty(S, (T, \|\,.\,\|), \Omega, u) \subseteq \ell_\infty(S, (T, \|\,.\,\|), \Omega, v),$$

then $\liminf_s \left|\frac{u(s)}{v(s)}\right| > 0$.

Proof:

Assume that the

$$\ell_\infty(S, (T, \|\,.\,\|), \Omega, u) \subseteq \ell_\infty(S, (T, \|\,.\,\|), \Omega, v)$$

holds but

$$\liminf_s \left|\frac{u(s)}{v(s)}\right| = 0.$$

Then there exists a sequence $< s_k >$ in S of distinct points such that for each $k \ge 1$, we have

(3.5) $k\,|u(s_k)| < |v(s_k)|$

We now choose $t \in T$ and $\|t\| = 1$ and define $\phi : S \to T$ by

$$\text{(3.6)}\quad \phi(s) = \begin{cases} (u(s_k))^{-1}\, t, & \text{for } s = s_k,\ k = 1, 2, 3, \ldots, \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

Let $r > 0$. Then we have

$$\sup_{s\in S} \Omega\left(\frac{\| u(s)\,\phi(s)\|}{r}\right) = \sup_{k\ge 1} \Omega\left(\frac{\| u(s_k)\,\phi(s_k)\|}{r}\right)$$
$$= \Omega\left(\frac{\|t\|}{r}\right)$$
$$= \Omega\left(\frac{1}{r}\right)$$
$$< \infty$$

This clearly shows that $\phi \in \ell_\infty(S, (T, \|\,.\,\|), \Omega, u)$.

But on the other hand ,in view of (3.5) and (3.6) ,we have

$$\sup_{s\in S} \Omega\left(\frac{\| v(s)\,\phi(s)\|}{r}\right) = \sup_{k\ge 1} \Omega\left(\frac{\| v(s_k)\,\phi(s_k)\|}{r}\right)$$
$$= \sup_{k\ge 1} \Omega\left(\frac{1}{r}\left|\frac{v(s_k)}{u(s_k)}\right| \|u\|\right)$$
$$\ge \sup_{k\ge 1} \Omega\left(\frac{k}{r}\right)$$
$$= \infty.$$

This shows that $\phi \notin \ell_\infty(S, (T, \|\,.\,\|), \Omega, v)$, a contradiction. This completes the proof.

When the Theorems 3.5 and 3.6 are combined, we get :

Theorem 3.7: For any $u, v \in N(S, \mathbf{C} - \{0\})$,

$$\ell_\infty(S, (T, \|\,.\,\|), \Omega, u) \subseteq \ell_\infty(S, (T, \|\,.\,\|), \Omega, v)$$

if and only if

$$\liminf_s \left|\frac{u(s)}{v(s)}\right| > 0.$$

Corollary 3.8: Let $u \in N(S, \mathbf{C} - \{0\})$. Then

$$\ell_\infty(S, (T, \|\,.\,\|), \Omega, u) \subseteq \ell_\infty(S, (T, \|\,.\,\|), \Omega)$$

if and only if $\liminf_s |u(s)| > 0$.

Proof:

By considering the function v on S such that

$$v(s) = 1 \text{ for all } s \in S$$

in Theorem 3.7 , one can easily obtain the assertion.

In the following example, we show that inspite of satisfaction of the condition $\liminf_s \left|\frac{u(s)}{v(s)}\right| > 0$ of the Theorem 3.7, $\ell_\infty(S, (T, \|\,.\,\|), \Omega, u)$ may strictly be contained in $\ell_\infty(S, (T, \|\,.\,\|), \Omega, v)$.

Example 3.9:

Let S be any set and $< s_k >$ be a sequence of distinct points of S. Take $t \in T$ such that $\|t\| = 1$ and define $\phi : S \to T$ by

$$\text{(3.7)}\quad \phi(s) = \begin{cases} t, & \text{if } s = s_k,\ k = 1, 2, 3, \ldots, \text{and} \\ \theta, & \text{otherwise.} \end{cases}$$

Further if $s = s_k$, $u(s_k) = k$, $v(s_k) = 1$ for all values of k and $u(s) = 3$, $v(s) = 2$ otherwise

Then for $s = s_k$ and $k \ge 1$, we have

$$\left|\frac{u(s)}{v(s)}\right| = k \text{ for all values of } k\,,$$

and

$$\left|\frac{u(s)}{v(s)}\right| = \frac{3}{2} \text{ otherwise.}$$

Therefore $\liminf_s \left|\frac{u(s)}{v(s)}\right| > 0$.

But taking $r > 0$ we see below that

$$\sup_{s\in S} \Omega\left(\frac{\| v(s)\,\phi(s)\|}{r}\right) = \sup_{k\ge 1} \Omega\left(\frac{\| v(s_k)\,\phi(s_k)\|}{r}\right)$$
$$= \Omega\left(\frac{\|t\|}{r}\right)$$
$$= \Omega\left(\frac{1}{r}\right)$$
$$< \infty.$$

This shows that $\phi \in \ell_\infty(S, (T, \|\,.\,\|), \Omega, v)$. But on the other hand in view of (3.7), we have

$$\sup_{k\ge 1} \Omega\left(\frac{\| u(s_k)\,\phi(s_k)\|}{r}\right) = \sup_{k\ge 1} \Omega\left(\frac{\| k\,t\|}{r}\right)$$
$$= \Omega\left(\frac{k}{r}\right)$$
$$= \infty,$$

implies that $\phi \notin \ell_\infty(S, (T, \|\,.\,\|), \Omega, u)$.

This shows that inspite of the satisfaction of the condition of Theorem 3.7 , $\ell_\infty(S, (T, \|\,.\,\|), \Omega, u)$ is strictly contained in $\ell_\infty(S, (T, \|\,.\,\|), \Omega, v)$.

Theorem 3.10: Let u, v $\in$ N (S, **C** - {0}), ℓ_∞ (S, (T, ‖ . ‖), Ω, v) $\subseteq$ ℓ_∞ (S, (T, ‖ . ‖) , Ω, u)

$$\text{if } \lim\sup_s \left|\frac{u(s)}{v(s)}\right| < \infty.$$

Proof:

Assume that

$$\lim\sup_s \left|\frac{u(s)}{v(s)}\right| < \infty,$$

then there exists a constant d > 0 such that

$$|u(s)| < d\,|v(s)|$$

for all but finitely many s $\in$ S.

Let $\phi \in \ell_\infty$ (S, (T, ‖ . ‖) , Ω, v) and $r_1 > 0$ is associated with ϕ . Then

$$(3.8)\qquad \sup_{s\in S} \Omega\left(\frac{\|v(s)\,\phi(s)\|}{r_1}\right) < \infty.$$

Let us choose r > 0 such that $d\, r_1 \le r$, then for such r , using non decreasing property of Orlicz function Ω we have

$$\Omega\left(\frac{\|u(s)\,\phi(s)\|}{r}\right) \le \Omega\left(\frac{|u(s)|\,\|\phi(s)\|}{r}\right)$$
$$\le \Omega\left(\frac{d\,|v(s)|\,\|\phi(s)\|}{r}\right)$$
$$\le \Omega\left(\frac{\|v(s)\,\phi(s)\|}{r_1}\right)$$

and hence in view of (3.8),

$$\sup_{s\in S} \Omega\left(\frac{\|u(s)\,\phi(s)\|}{r}\right) \le \sup_{s\in S} \Omega\left(\frac{\|v(s)\,\phi(s)\|}{r_1}\right) < \infty.$$

This shows that $\phi \in \ell_\infty$ (S, (T, ‖ . ‖), Ω, u) and hence

ℓ_∞ (S, (T, ‖ . ‖), Ω, v) $\subseteq$ ℓ_∞ (S, (T, ‖ . ‖), Ω, u).

This completes the proof.

Theorem 3.11: Let u, v $\in$ N (S, **C** - {0}), ℓ_∞ (S, (T, ‖ . ‖), Ω, v) $\subseteq$ ℓ_∞ (S, (T, ‖ . ‖) , Ω, u), then $\lim\sup_s \left|\frac{u(s)}{v(s)}\right| < \infty$.

Proof:

Assume that

ℓ_∞ (S, (T, ‖ . ‖) , Ω, v) $\subseteq$ ℓ_∞ (S, (T, ‖ . ‖), Ω, u)

but

$$\lim\sup_s \left|\frac{u(s)}{v(s)}\right| = \infty.$$

Then we can find a sequence $< s_k >$ of distinct points in S such that for each $k \ge 1$,

(3.9) $|u(s_k)| > k\,|v(s_k)|$.

We now choose t $\in$ T and ‖ t ‖ = 1 and define $\phi : S \to T$ by

$$(3.10)\quad \phi(s) = \begin{cases} (v(s_k))^{-1}\, t, & \text{for } s = s_k,\ k = 1, 2, 3, \ldots, \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

Let r > 0. Then we have

$$\sup_{s\in S} \Omega\left(\frac{\|v(s)\,\phi(s)\|}{r}\right) = \sup_{k\ge 1} \Omega\left(\frac{\|v(s_k)\,\phi(s_k)\|}{r}\right)$$
$$= \Omega\left(\frac{\|t\|}{r}\right)$$
$$= \Omega\left(\frac{1}{r}\right)$$
$$< \infty .$$

This shows that $\phi \in \ell_\infty$ (S, (T, ‖ . ‖) , Ω, v).

But on the other hand in view of (3.9) and (3.10) ,we have

$$\sup_{s\in S} \Omega\left(\frac{\|u(s)\,\phi(s)\|}{r}\right) = \sup_{k\ge 1} \Omega\left(\frac{\|u(s_k)\,(v(s_k))^{-1}\,t\|}{r}\right)$$
$$= \sup_{k\ge 1} \Omega\left(\frac{\left|\frac{u(s_k)}{v(s_k)}\right| \|t\|}{r}\right)$$
$$\ge \sup_{k\ge 1} \Omega\left(\frac{k}{r}\right)$$
$$= \infty,$$

implies that $\phi \notin \ell_\infty$ (S, (T, ‖ . ‖) , Ω, u). This leads to a contradiction and completes the proof.

When the Theorems 3.10 and 3.11 are combined, we get :

Theorem 3.12: Let u, v $\in$ N (S, **C** - {0}), ℓ_∞ (S, (T, ‖ . ‖), Ω, v) $\subseteq$ ℓ_∞ (S, (T, ‖ . ‖) , Ω, u) if and only if

$$\lim\sup_s \left|\frac{u(s)}{v(s)}\right| < \infty.$$

Corollary 3.13: Let u $\in$ N (S, **C** - {0}). Then ℓ_∞ (S, (T, ‖ . ‖) , Ω) $\subseteq$ ℓ_∞ (S, (T, ‖ . ‖) , Ω, u) if and only if $\lim\sup_s |u(s)| < \infty$; and

Proof:

By considering the function v on S such that

$$v(s) = 1 \text{ for all } s \in S$$

in Theorems 3.12 , one can easily obtain the assertion.

If Theorems 3.7 and 3.12 are combined, we get :

Theorem 3.13: If u, v $\in$ N (S, **C** - {0}), then ℓ_∞ (S, (T, ‖ . ‖) , Ω, u) = ℓ_∞ (S, (T, ‖ . ‖) , Ω, v) if and only if

$$0 < \lim\inf_s \left|\frac{u(s)}{v(s)}\right| \le \lim\sup_s \left|\frac{u(s)}{v(s)}\right| < \infty.$$

Corollary 3.14: Let u $\in$ N (S, **C** - {0}). Then ℓ_∞ (S, (T, ‖ . ‖) , Ω) = ℓ_∞ (S, (T, ‖ . ‖) , u) if and only if

$$0 < \lim\inf_s |u(s)| \le \lim\sup_s |u(s)| < \infty.$$

Proof:

By considering the function v on S such that

$$v(s) = 1 \text{ for all } s \in S$$

in Theorem 3.13, one can easily obtain the assertion.

4. Conclusion

This paper establishes some of the results that characterize the linear structures of the class ℓ (S, (T, ‖. ‖),Ω, u) of normed space valued function space using Orlicz function. In fact, these results can be used for further generalization and unification to investigate the properties of the various existing Orlicz function spaces studied in Functional Analysis.

References

[1] Bhardwaj,V.N.and Bala,I.**:** Banach space valued sequence space ℓ_M (X,p),Int. J. of Pure and Appl. Maths., 41(5) (2007), 617–626.

[2] Chen, S.T.: Geometry of Orlicz Spaces, Dissertations Math. , The Institute of Mathematics, Polish Academy of Sciences, (1996).

[3] Ghosh, D. and Srivastava P.D.**:** On Some Vector Valued Sequence Spaces using Orlicz Function, Glasnik Matematicki, 34 (54) (1999), 253–261.

[4] Kamthan, P.K. and Gupta, M.: Sequence Spaces and Series, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 65,(1981).

[5] Khan, V.A.**:** On a new sequence space defined by Orlicz functions; Common. Fac. Sci. Univ. Ank-series; 57(2) (2008), 25–33

[6] Kolk,E.**:** Topologies in generalized Orlicz sequence spaces, Filomat, 25(4) (2011),191-211

[7] Krasnosel'skiî, M.A. and Rutickiî, Y.B.: Convex Functions and Orlicz Spaces,P. Noordhoff Ltd-Groningen-The Netherlands, (1961)

[8] Lindenstrauss,J. and Tzafriri, L.:Classical Banach spaces; Springer-Verlag, New York/Berlin,(1977)

[9] Maddox, I.J.**:** Some properties of paranormed sequence spaces; London. J. Math. Soc. ,2(1) (1969) , 316-322

[10] Pahari,N.P., **:** On Banach Space Valued Sequence Space l_∞ (X,M, $\bar{\lambda}$, $\bar{p}$, L) Defined by Orlicz Function , Nepal Jour of Science and Tech.,12 (2011), 252-259

[11] Pahari,N.P.,: On Certain Topological Structures of Paranormed Orlicz Space (S ((X ,‖.‖)**,**Φ, $\bar{\alpha}$, $\bar{u}$), F) of Vector Valued Sequences**,** International Jour. of Mathematical Archive , 4 (11) (2013), 231-241.

[12] Parashar, S.D. and Choudhary, B.**:** Sequence spaces defined by Orlicz functions, Indian J. Pure Appl. Maths., 25(4) (1994), 419–428.

[13] Rao, K. C. & Subremanina, N.: The Orlicz Space of Entire Sequences, IJMass, (2004), 3755-3764.

[14] Rao, M. M. and Ren,Z. D. **:** Theory of Orlicz spaces , Marcel Dekker Inc., New York, (1991).

[15] Savas E. and Patterson, F.: An Orlicz Extension of Some New Sequence Spaces, Rend. Instit. Mat. Univ. Trieste, 37 (2005), 145–154.

[16] Srivastava,J.K and Pahari,N.P.**:** On Banach space valued sequence space l_M (X, $\bar{\lambda}$, $\bar{p}$, L) defined by Orlicz function, South East Asian J.Math. & Math. Sc., 10(1) (2011), 39-49.

[17] Srivastava,J.K and Pahari,N.P. **:** On Banach space valued sequence space c_0 (X,M, $\bar{\lambda}$, $\bar{p}$, L) defined by Orlicz function, Jour. of Rajasthan Academy of Physical Sciences, 11(2) (2011), 103-116.

[18] Srivastava,J.K.and Pahari,N.P.**:** On 2-Banach space valued paranormed sequence space c_0 (X, M, ‖.,.‖, $\bar{\lambda}$, $\bar{p}$) defined by Orlicz function, Jour. of Rajasthan Academy of Physical Sciences,12(3)(2013), 319-338.

[19] Tiwari, R. K and Srivastava, J. K. **:** On certain Banach space valued function spaces- II, Math, Forum, 22 (2010), 1-14