International Journal of Scientific Engineering and Research (IJSER)
www.ijser.in
ISSN (Online): 2347-3878
Volume 2 Issue 9, September 2014

Friendship Two-Graphs in Route Mapping Technology

[1]S. Manimekalai, [2]S. Gokilamani, [3]R. Anandhi

Abstract: ***The aim of the paper is extending friendship graphs to two-graphs, a two-graph being an ordered pair $G = (G_0, G_1)$, every unordered pair of distinct vertices u, v is connected by a unique bicolored 2-path in route mapping. Construct an infinite such graph and this construction can be extended to an infinite (uncountable) family. Also finding a finite friendship two-graph, conjecture that is unique and proved this conjecture for the two-graphs that have a dominating vertex in route mapping.***

Keywords: l-friendship graphs, windmill graph, simple node, complex node, Kotzig generalized friendship graphs.

1. Introduction

A friendship graph is a graph in which every two distinct vertices have exactly one common neighbor. A friendship graph consists of triangles incident to a common vertex. Kotzig generalized friendship graphs to graphs in which every pair of vertices is connected by λ paths of length K. His conjecture is that for $K \geq 3$, there is no finite graph in which every pair of vertices is connected by a unique path.

The Friendship Theorem traces its roots back to the relatively early days of graph theory. Most authors recognize that the first published proof was given by Erdos, Renyi and Sos in 1966 in a Hungarian journal. Theorem (Erdos): If G_n is a graph in which any two points are connected by a path of length 2 and which does not contain any cycle of length 4, then n = 2k+1 and G_n consists of k triangles which have one common vertex. Theorem (Huneke): If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others. Wilf gave a proof in 1969 with roots in linear algebra and projective geometry. He computes the eigenvalues of the incidence matrix of the graph and uses this to produce a contradiction. This becomes a common way to prove the friendship theorem. Although Wilf's is unique that starts by delving into geometry.

More recently J. M. Hammersley provided a proof at a conference in 1983 that avoided using eigenvalues but involved admittedly complicated numerical techniques. Hammersley also extends the friendship theorem in what he calls the "love problem". Friendship is usually taken to be irreflexive, but love, as he points out, can be narcissistic and hence a reflexive relation. In 1999, Aigner and Ziegler immortalized the friendship theorem in proofs from the book and covering the greatest theorems of all time. In his undergraduate textbook, Introduction to Graph Theory, D. B. West, 2001, the proof similar to Longyear and Parson's that counts common neighbors of vertices and cycles is included.

2. Preliminary Notes

Definition 2.1: Every pair of nodes has exactly l common neighbors is called the l-friendship condition. The graphs that satisfy the l-friendship condition are exactly the $P_l(2)$-graphs and they are called l-friendship graphs.

Definition 2.2: A graph is called a windmill graph if it consists of $k \geq 1$ triangles which have a unique common node known as the politician. Clearly any windmill graph is a friendship graph.

Definition 2.3: In a friendship graph G every node v with $\deg(v) = 2$ is called a simple node. Otherwise it is called complex node.

Definition 2.4: A two-graph (G_0, G_1) is called a friendship two-graph if for every unordered pair of distinct vertices u, v there exists a unique bicolored 2-path connecting u and v.

Definition 2.5: G is called k-connected if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$.

Definition 2.6: An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a chord of the cycle.

3. Friendship Two-Graphs and Its Combinatorial Results

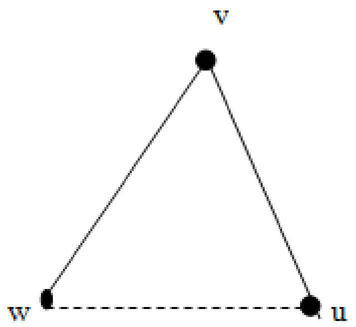
In this section a purely combinatorial proof of the friendship theorem is proposed. That is every friendship graph is a windmill graph. Denote by C_4 a node-simple cycle on 4 nodes, by $N(v)$ the set of neighbors of v in G and $N[v] = N(v) \cup \{v\}$.

Theorem 3.1
Every non-trivial finite friendship two-graph $G = (G_0, G_1)$ has minimum degree at least three.

ISSN (Online): 2347-3878
Volume 2 Issue 9, September 2014

Proof

Let $G = (G_0, G_1)$ be a non-trivial friendship two-graph having a vertex v of degree at most two. Clearly v cannot be an isolated vertex, so assume that v is 1-adjacent to a vertex w. Consider the unique bicolored 2-path (v, u, w) connecting v and w. If uv is a 0-edge then G does not have a bicolored 2-path connecting u and v, since v has degree at most two. Thus uv is a 1-edge and uw is a 0-edge, which is illustrated by the following figure.



The subgraph induced by the set $\{u, v, w\}$.

Hence a (2+2) - cycle is a 4-cycle that contains exactly two 0-edges and exactly two 1-edges.

Theorem 3.2

i) The only 1-edge incident to u is uv.

ii) The only 1-edges incident to w are vw and wx.

Proof

i) Suppose that uz is a 1-edge with $z \neq v$. Then for every vertex $z \neq v$, exactly one of uz or wz is a 0-edge, w and z are 0-adjacent. To obtain a (2+2) –cycle (u, z, w, x) leading to a contraction to a friendship two-graph does not have (2+2)-cycles. Hence the only 1-edge incident to u is uv.

ii) Now let wz be a 1-edge with $z \neq v, x$. Then for every vertex $z \neq v$, exactly one of uz or wz is a 0-edge, u and z are 0-adjacent and (w, z, u, x) is a (2+2) -cycle leading to a contradiction to a friendship two-graph does not have (2+2) -cycles. Hence the only 1-edges incident to w are vw and wx.

Property 3.3

The only 1-edges incident to x are wx and xy.

Proof

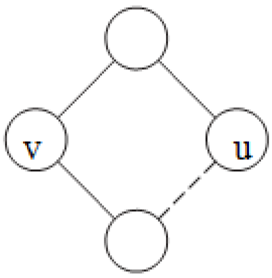
Suppose that xz is a 1-edge with $z \neq w, y$. If uz is a 0-edge, then (u, w, x, z) is a (2+2)-cycle which is a contradiction that friendship two-graph does not have (2+2) -cycles. Then for every vertex $z \neq v$, exactly one of uz or wz is a 0-edge, w and z are 0-adjacent. But then (w, y, x, z) is a (2+2)-cycle which is a contradiction that friendship two-graph does not have (2+2)-cycles. Hence the only 1-edges incident to x are wx and xy.

Proposition 3.4

A friendship graph G contains no C_4 as a subgraph as well as the distance between any two nodes in G is at most two.

Proof

If G includes C_4 as a subgraph, then there are two nodes v and u with at least two common neighbors as it is illustrated in figure. This is in contradiction to the friendship condition. On the other hand, if a pair (v, u) of G has distance at least three, then v and u have no common neighbor in G which is also a contradiction.

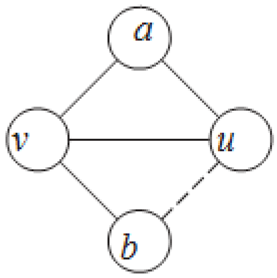


Lemma 3.5

For every node v of a friendship graph G, $N[v]$ induces a windmill graph.

Proof

Consider two nodes v and $u \in N(v)$. Assuming that they have a unique common neighbor a, as it is illustrated in figure.



Consider now another node $b \in N(v) \backslash \{u, a\}$. If $b \in N(u)$, then G includes a C_4 as a subgraph which is a contradiction due to a friendship graph G contains no C_4 as a subgraph as well as the distance between any two nodes in G is at most two. Thus, $b \notin N(v)$ produces

International Journal of Scientific Engineering and Research (IJSER)
www.ijser.in
ISSN (Online): 2347-3878
Volume 2 Issue 9, September 2014

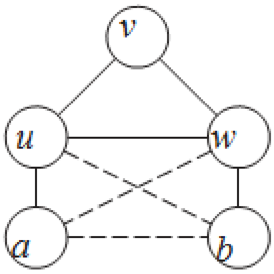
with v exactly one triangle. Therefore for every node v of G, $N[v]$ induces a windmill graph.

Lemma 3.6

If a friendship graph G has at least one simple node then G is a windmill graph.

Proof

Consider a simple node v of G with $N(v) = \{u, w\}$ as it is illustrated in figure.



Due to for every node v of a friendship graph G, $N[v]$ induces a windmill graph. So that u and w are also neighbors. At first since u and w have a unique common neighbor all their neighbors are distinct except v. In this case where G is constituted of only these three nodes G is obviously a windmill graph. Otherwise every other node of $V\setminus\{v, u, w\}$ is either neighbor of u or of w, since in the opposite case it would have no common neighbor with v which is a contradiction.

Finally consider two nodes $a \in N(u)\setminus\{v, w\}$ and $b \in N(w)\setminus\{v, u\}$. Then a and b are not neighbors, since otherwise u, w, b and a would induce a C_4, which is a contradiction to a friendship graph G contains no C_4 as a subgraph as well as the distance between any two nodes in G is at most two. It follows that the distance between a and b is three which is also a contradiction. Thus at least one node of $\{u, w\}$ is simple and the other one is neighbored to all other nodes in G. It follows that G is a windmill graph due to for every node v of a friendship graph G, $N[v]$ induces a windmill graph.

Lemma 3.7

There is at least one simple node in any friendship graph G.

Proof

The proof proceeded by contradiction. Suppose that all nodes of G are complex that is their degree is greater than two. Then by if a friendship graph G has no simple node then G is a regular graph. This implies that G is a $2k$ −regular graph with $n = 2k(2k-1)+1$ nodes for some $k \geq 2$. For an arbitrary natural number $l \geq 1$. Let $T(l)$ be the set of all ordered l −tuples $< v_1, v_2, \ldots, v_l >$ of nodes of G such that v_i is neighbored with v_{i+1} for every $i \in \{1,2,\ldots\ldots,l-1\}$. Since $n = 2k(2k-1)+1$, it holds that

$$|T(l)| = n.(2k)^{l-1} \equiv 1 \bmod (2k-1) \quad 3.1$$

for every $l \geq 1$. If the nodes v_l and v_1 are neighbored then the tuple $< v_1, v_2, \ldots, v_l >$ constitutes a closed l − walk in G. Let $C(l) \subseteq T(l)$ be the set of all closed l −walks. Let furthermore $C^*(l) = \{< v_1, v_2, \ldots, v_l > \in T(l): v_l = v_1\}$ be the set of all closed $(l-1)$ −walks in G.

Consider now the surjective mapping $f: C(l) \to T(l-1)$ such that $f(< v_1, v_2, \ldots, v_l >) = < v_1, v_2, \ldots, v_{l-1}, v_l >$. For every tuple $< v_1, v_2, \ldots, v_{l-1} >$ of $T(l-1)\setminus C^*(l-1)$, that is with $v_{l-1} \neq v_1$ it holds that $< v_1, v_2, \ldots, v_{l-1} > = f\,(< v_1, v_2, \ldots, v_{l-1}, y >)$ where y is the unique common neighbor of v_{l-1} and v_1 in G. On the other hand, for every tuple $< v_1, v_2, \ldots, v_{l-1} = v_1 >$ of $C^*(l-1)$ it holds that $< v_1, v_2, \ldots, v_{l-1} = v_1 > = f(< v_1, v_2, \ldots, v_{l-1} = v_1, z$ where z is any of the $2k$ neighbors of v_1 in G. Since f is surjective and due to (4.3.1) it follows that

$$|C(l)| = 2k\,|C^*(l-1)| + |T(l-1) \setminus C^*(l-1)|$$

$$\equiv |T(l-1)| \bmod (2k-1) \quad 3.2$$

$\equiv 1 \bmod (2k-1)$, for every $l \geq 2$.

Now for an arbitrary prime divisor p of $2k-1$. Consider the bijective mapping $\pi: C(p) \to C(p)$ with $\pi\,(< v_1, v_2, \ldots, v_p >) = < v_2, v_3, \ldots v_p, v_1 >$. Since p is a prime number all tuples $\pi^i(< v_1, v_2, \ldots, v_p >)$ where $i \in \{1,2,\ldots\ldots,p\}$ are distinct. The mapping π defines in a trivial way an equivalence relation: the tuples $< v_1, v_2, \ldots, v_p >$ and $< w_1, w_2, \ldots, w_p >$ are equivalent if there is a number $t \in \{1,2,\ldots\ldots,p\}$ such that $\pi^t(< v_1, v_2, \ldots, v_p >) = < w_1, w_2, \ldots, w_p >$. This equivalence relation partitions $C(p)$ into equivalence classes of p elements each and thus, it holds that

$$C(p) \equiv 0 \bmod (p) \quad 3.3$$

Since p is a prime divisor of $2k-1$, equation (3.3) is in contradiction to (3.2) for $l = p$. Thus G is not a

$2k$ –regular graph and therefore it has at least one simple node.

4. Example

A graph consists of a set of nodes connected by edges. The original internet graph is the web itself where WebPages are nodes and links are edges. In social graphs, the nodes are people and the edges friendship. Edges are what mathematicians call as relations. Two important properties that relations can either have or not have are symmetry (if A ~ B then B ~ A) and transitivity (if A ~ B and B ~ C then A ~ C).

5. Conclusion

A friendship graph is a graph in which every two distinct vertices have exactly one common neighbor. All finite friendship graphs are known, each of them consists of triangles having a common vertex. There is no finite friendship two-graph with minimum vertex degree at most two. Generalized the simple friendship condition in a natural way to the $l-$ friendship condition:"Every pair of nodes has exactly $l \geq 2$ common neighbors" and proved that every graph which satisfies this condition is a regular graph for route mapping technology.

References

[1] Bondy, J.A.: Kotzig's conjecture on generalized friendship graphs-a survey, In: Cycles in Graphs (Burnaby, B.C., 1982). North-Holland Math. Stud., vol.115, pp. 351-366.North- Holland, Amsterdam (1985)
[2] Boros, E., Gurvich, V.: Vertex-and edge-minimal and locally minimal graphs, Discrete Math. 309(12), 3853-3865(2009)
[3] J.A. Bondy and U.S.R. Murty. Graph theory with applications. American, Elsevier Publ. Co., Inc., 1976
[4] Chvatal, V., Graham, R.L., Perold, A.F., Whitesides, S. H.: Combinatorial designs related to the strong perfect graph conjecture. Discrete Math. 26(2), 83-92(1979)
[5] Erdos, P., Renyi, A., Sos, V.T.: On a problem of graph theory. Studia Sci.Math.Hungar.1, 215-235(1966)
[6] Gurvich, V.: Decomposing complete edge –chromatic graphs and hypergraphs. Revisited, Discrete Appl.Math.157, 3069-3085(2009)
[7] Gurvich, V.: Some properties and applications of complete edge-chromatic graphs and hyper-graphs. Soviet Math. Dokl. 30(3), 803-807(1984)
[8] Katie Leonard. The friendship theorem and projective planes. Portland State University, December 7 2005
[9] A. Kotzig. Regularly k-path connected graphs. Congresus Numerantium, 40:137-141, 1983
[10] Kostochka A.V.: The nonexistence of certain generalized friendship graphs, In: Combinatorics (Eger, 1987). Colloq. Math. Soc. Janos Bolyai, vol. 52, North-Holland, Amsterdam (1988)
[11] A. Kotzig. Selected open problems in graph theory. Academic Press, New York
[12] A. Kostochka. The nonexistence of certain generalized friendship graphs. Combinatorics (Eger, 1987), Colloq. Math. Soc. J'anos Bolyai, 52:341-356, 1988
[13] D.B. West. Introduction to Graph Theory. Prentice Hall, 2 edition, 2001
[14] H.S. Wilf. The friendship theorem. Combinatorial mathematics and its applications, 1971