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# Derivatives of Divided Differences

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**Abstract:** ***Let V be a finite set containing k points.*** $f(V)$ ***is divided difference of*** $f$ ***at V. The m-th order Peano derivative of*** $f(\{x\}\cup V)$ ***is defined and is denoted by*** $f_m(x;V)$***. It is same as generalised divided diference which is defined in [1]. We have proved a decomposition theorem and a mean value theorem for this generalised divided diference. Also many of the properties are studied.***

## 1. Introduction

Suppose V be a finite set, $x \notin V$ and $f(V\cup\{x\})$ is divided difference of $f$. Let $f_m(x,V)$ is m-th order Peano derivative of $f(V\cup\{x\})$ regarded as a function of $x$ .In $[1]$ Fejzic, Svetic and Weil have termed the iterated limit of divided difference by generalized divided difference and use it to study the properties of n-convex functions. In this article we have studied the properties of that generalized divided difference, which is equivalent to $f_m(x,V)$. It is shown that $f_m(x;V)$ can be written as a sum of an n-th order Peano derivative of a function and n-th order divided difference of another function. For an n-convex function the properties of repeated limits of $f_m(x;V)$ are studied.

In [2] Mukhopadhayay and Ray, a mean value theorem for divided difference is proved. Here we have presented a mean value theorem for this generalized divided difference.

## 2. Definition and Notation

Let $f: E \to \Re$, $V=\{x_0,\ldots,x_n\}\subset E$ , then the divided difference of $f$ at $V$ is defined by

$$f(V)=f(x_0,\ldots,x_n)=\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)}$$

where

$$\omega(x)=\prod_{i=0}^{n}(x-x_i)$$

We write $f(x;V)$ instead of $f(V\cup\{x\})$.

Let $f: E\to\Re$, and let $n\in\mathrm{N}$. Then $f$ is n-convex in $E$ if for each subset V of $E$ containing n+1 points , $f(V)\geq 0$.

Let $x\in E-V$ be right hand limit point of $E$, then right Peano derivative of divided differences with respect to the set E is defined inductively as

$$f_1^+(x;V)=\lim_{\substack{y\to x^+\\ y\in E}}\frac{f(y;V)-f(x;V)}{(y-x)}$$

and if $f_r^+(x;V)$ exist for $1\leq r<m$ then m-th order derivative

$$f_m^+(x;V)=\lim_{\substack{y\to x^+\\ y\in E}}\gamma_m(f,x,y,V)$$

where

$$\gamma_m(f,x,y,V)=\frac{m!}{(y-x)^m}\left\{f(y;V)-\sum_{i=0}^{m-1}\frac{(y-x)^i}{i!}f_i^+(x;V)\right\}.$$

Here we assume $f_0^+(x;V)=f(x;V)$.

If $V=\phi$, then we write $\gamma_m(f,x,y)$ instead of $\gamma_m(f,x,y,\phi)$. In this case $f_m^+(x;V)$ is the usual right hand Peano derivative of $f$ at $x$ of order m and is denoted by $f_m^+(x)$. If $x\in E-V$ be a left hand limit point. We define left hand Peano derivative of divided difference $f_m^-(x;V)$ in similar way. If $x$ is both sided limit point and $f_m^+(x;V)$ and $f_m^-(x;V)$ both exist and $f_i^+(x;V)=f_i^-(x;V)$ for $i$ =1, 2 ,..., m, then $f$ is said to have the m-th order Peano derivative of divided difference $f_m(x;V)$. Clearly for a fixed V$\subset$E, $f_m(x;V)$ is the m-th order Peano derivative of the function $f(x;V)$, regarded as a function of $x$. Hence for $x\notin V$, $f_m(x;V)$ exists if and only if $f_m(x)$ exists.

Let $f:E\to\Re$ and $V\subset E$ be finite and $x\in E-V$ be a limit point of E. Let $\{x_1,\ldots,x_k\}$ be k distinct points in $E-V$. Then we define

$$[f,x,V]^k=\lim_{\substack{x_k\to x\\ x_k\in E}}\cdots\lim_{\substack{x_1\to x\\ x_1\in E}} f(x,x_1,\ldots,x_k;V).$$

In [1], $[f,x,V]^k$ is termed as generalized divided difference of order k. Clearly it is same as $f_k(x;V)$. In what follows from now we shall drop $y\in E$ under the limit notation.

## 3. Properties of $f_n(x;V)$

**Theorem 3.1** *Let* $f:E\to\Re$ *and* $V=\{x_0,x_1,\ldots,x_n\}\subset E$. *Let* $f_m$ *exists on E and* $u\in E-V$ *be both sided limit point of E, then*

$$f_m(u;V)=\phi_m(u)+m!\psi(V)$$

where $\phi(t)=\dfrac{f(t)}{\omega(t)}$ and $\psi(t)=\dfrac{f(t)}{(t-u)^{(m+1)}}$

*Proof.* We prove the theorem by induction on $\gamma_m(f,u,y,V)$. We prove

$$(1)\quad \gamma_m(f,u,y,V)=\gamma_m(\phi,u,y)+m!\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)^m(x_i-y)}$$

Clearly

$$\gamma_1(f,u,y,V)=\frac{f(y;V)-f(u;V)}{y-u}$$

$$=f(u,y;V)$$

$$=\frac{f(u)}{(u-y)\omega(u)}+\frac{f(y)}{(y-u)\omega(y)}+\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)(x_i-y)}$$

$$=\frac{\phi(y)-\phi(u)}{y-u}+\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)(x_i-y)}$$

$$=\gamma_1(\phi,u,y)+\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)(x_i-y)}$$

So (1) is true for m=1. Let it is true for $m=r\geq 1$. So putting m=r in (1) and letting $y\to u, y\in E$, since $f_r$ exists we get

$$f_r(u;V)=\phi_r(u)+r!\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)^{r+1}}$$

Now, $\gamma_{r+1}(f,u,y,V)=\dfrac{(r+1)!}{(y-u)^{r+1}}\left\{f(y;V)-\sum_{i=0}^{r}\dfrac{(y-u)^i}{i!}f_i(u;V)\right\}$

$$=\frac{(r+1)}{(y-u)}\left[\frac{r!}{(y-u)^r}\left\{f(y;V)-\sum_{i=0}^{r-1}\frac{(y-u)^i}{i!}f_i(u;V)\right\}-f_r(u;V)\right]$$

$$=\frac{(r+1)}{(y-u)}[\gamma_r(f,u,y,V)-f_r(u;V)]$$

$$=\frac{(r+1)}{(y-u)}\left[\{\gamma_r(\phi,u,y)+r!\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)^r(x_i-y)}\}-\{\phi_r(u)+r!\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)^{r+1}}\}\right]$$

$$=\frac{(r+1)}{(y-u)}\left[\{\gamma_r(\phi,u,y)-\phi_r(u)\}+r!\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)^r}\{\frac{1}{x_i-y}-\frac{1}{x_i-u}\}\right]$$

$$=\gamma_{r+1}(\phi,u,y)+(r+1)!\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)^{r+1}(x_i-y)}$$

Hence (1) is true for m=r+1. So by induction (1) is true, now letting $y\to u$ we get

$$f_m(u;V)=\phi_m(u)+m!\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)(x_i-u)^{m+1}}$$
$$=\phi_m(u)+m!\psi(V)$$

**Theorem 3.2** *Suppose* $f:E\to\Re$ *and* $V\subset E$ *be finite set and* $x\in E-V$*, Then* $f_k(x;V)$ *is divided difference of* $f_k(x;\{t\})$ *as a function of* $t$.

*Proof.* Let $V=\{x_0,x_1,...,x_n\}$ , from Theorem 3.1 we get

$$f_k(x;V-\{x_0\})=\left[\frac{f(x)(x-x_0)}{w(x)}\right]_k+k!\psi(V-\{x_0\})$$

and

$$f_k(x;V-\{x_1\})=\left[\frac{f(x)(x-x_1)}{w(x)}\right]_k+k!\psi(V-\{x_1\})$$

The suffix k denotes the k-th order Peano derivative of the expression in bracket. Hence

$$\frac{f_k(x;V-\{x_0\})-f_k(x;V-\{x_1\})}{x_1-x_0}=$$

$$\frac{1}{x_1-x_0}\left[\frac{f(x)(x-x_0)}{w(x)}-\frac{f(x)(x-x_1)}{w(x)}\right]_k+k!\frac{\psi(V-\{x_0\})-\psi(V-\{x_1\})}{x_1-x_0}$$

$$=\left[\frac{f(x)}{w(x)}\right]_k+k!\psi(V)$$

$= f_k(x;V)$

.

**Theorem 3.3** *Suppose* $f : E \to \Re$ *be continuous and* $V \subset E$ *be a finite set and* $x \in E - V$ *be right hand limit point of E. Now if* $f_{k+1}^{+}(x;V)$ *exists then*

$$(r+1)\lim_{u \to x^+} f_r^{+}(x;V \cup \{u\}) = f_{r+1}^{+}(x;V)$$

*Proof.* As in Newton divided difference interpolation formula we can write

$$f(u;V) = f(x_0;V) + (u - x_0) f(x_0, x_1;V) + ... + (u - x_0)...(u - x_{r-1}) f(x_0,..., x_r;V) +$$

$$(u - x_0)...(u - x_r) f(u, x_0,..., x_r;V)$$

Now performing the limit $\lim_{x_r \to x^+} ... \lim_{x_0 \to x^+}$ on both sides we get

$$f(u;V) = f(x;V) + (u - x) f_1^{+}(x;V) + ... + \frac{(u-x)^r}{r!} f_r^{+}(x;V) + \frac{(u-x)^{r+1}}{r!} f_r^{+}(x;V \cup \{u\})$$

So

$$(r+1) f_r^{+}(x;V \cup \{u\}) = \frac{f(u;V) - f(x;V) - \sum_{i=1}^{r} \frac{(u-x)^i}{i!} f_i^{+}(x;V)}{\frac{(u-x)^{r+1}}{(r+1)!}}$$

Now $u \to x^+$, we get the following result.

**Remark 3.4** *Theorem 3.4 holds for left derivative also.*

**Theorem 3.5** *Let* $r, m, n, s \in \mathrm{N}$ *and* $f$ *is n-convex on E,* $r + m + s < n$ *then*

$$\lim_{y_m \to x} ... \lim_{y_1 \to x} \lim_{x_s \to x} ... \lim_{x_1 \to x} m!(s-1)! f_r(x_1, x_2,..., x_s;\{y_1, y_2,..., y_m\}) = f_{r+(s-1)+m}(x)$$

If $r + m + s = n$, then at the points $x \in E$ where $f_{n-1}$ exists ( by Corollary 6.7 in [1] it is except a countable set )

$$\lim_{y_m \to x} ... \lim_{y_1 \to x} \lim_{x_s \to x} ... \lim_{x_1 \to x} m!(s-1)! f_{n-(s+m)}(x_1, x_2,..., x_s;\{y_1, y_2,..., y_m\}) = f_{n-1}(x)$$

*Proof.* Suppose $V = \{y_1, y_2,..., y_m\}$. Since $f$ is n-convex, the divided difference $f(x_1, x_2,..., x_{n-m}, y_1, y_2,..., y_m)$ is nondecreasing function of each $x_i$. so $f(x;V)$ is (n-m) convex. Suppose $\phi(x) = f(x;V)$. So $\phi$ is (n-m) convex. By Corollary 6.7 in [1], $[\phi_r(x)]_{s-1} = \phi_{s+r-1}(x)$ for $r + s < n - m$. So $[f_r(x;V)]_{s-1} = f_{r+s-1}(x;V)$.

$$\lim_{x_s \to x} ... \lim_{x_1 \to x} (s-1)! f_r(x_1, x_2,..., x_s;V) = [f_r(x;V)]_{s-1}$$

Also from Theorem and Remark, $\lim_{y_m \to x} ... \lim_{y_1 \to x} m! f_{r+s-1}(x;V) = f_{r+s+m-1}(x)$

This completes the proof for first part

If $r + s + m = n$, then also except some countable points $x$, $[\phi_r(x)]_{s-1} = \phi_{s+r-1}(x)$ for $r + s = n - m$. So as in first part we get the result replacing r by n-(m+s).

**Theorem 3.6** *Suppose* $V$ *and* $W$ *are finite subsets of* $[a,b]$ *with* $\mathrm{P}V\mathrm{P} = \mathrm{P}W\mathrm{P} = n$ *and* $f_k$ *exists on* $[a,b]$. *Let* $f_k(x;V) \le 0 \le f_k(x;W)$ *for some* $x \in [a,b]$. *Then there is* $A \subset [a,b] - \{x\}, \mathrm{P}A\mathrm{P} = n$, *such that* $\inf[V \cup W] \le y \le \sup[V \cup W], \forall y \in A,$ *and* $f_k(x; A) = 0$.

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*Proof.* From Theorem 3.1 we have $f_k(x;\{t\}) = [\frac{f(x)}{x-t}]_k + k!\frac{f(t)}{(x-t)^2}$. So if $g(t) = f_k(x;\{t\}),$ then $g$ has Darboux property in $[a,b]-\{x\}$. Now by Theorem 3.3 , $f_k(x;V)$ is divided difference of $g$ at the points of V . Hence by Theorem 3.1 of [2] the result follows.

## References

[1] Fejzic H, Svetic R E and Weil C E (2010) : Differentiation of n-convex functions . *Fund. Math.* 209 ,No.1 9-25 Zbl 1202.26008

[2] Mukhopadhayay S N, Ray S (2009) : Mean value Theorems for divided differences and approximate Peano derivatives. *Mathematica Bohemica* 134 No. 2, 165-171. Zbl 1212.26075