Finite Difference Approximation for First-Order Hyperbolic Partial Differential Equation Arising in Neuronal Variability with Shifts

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Abstract: ***This paper studies some finite difference approximations to find the numerical solution of first-order hyperbolic partial differential equation of mixed type. We are interested in the challenging issues in neuronal science stemming from the modeling of neuronal variability based on Stein's Model [8]. The resulting mathematical model is a first order hyperbolic partial differential equation having point-wise delay and advance which models the distribution of time intervals between successive neuronal firings. We construct, analyze and implement explicit numerical scheme for solving such type of initial and boundary-interval problems. Analysis shows that numerical scheme is conditionally stable, consistent and convergent in discrete L^∞ norm. Some numerical tests are reported to validate the computational efficiency of the numerical approximation.***

Keywords: hyperbolic partial differential equation, neuronal firing, point-wise delay and advance, finite difference method, Lax-Friedrichs scheme

1. Introduction

Partial differential-difference equations or more generally partial functional differential equations are of great importance, since they arise in many mathematical models of control theory, mathematical biology, climate models, mathematical economics, meteorology and many other areas, see [4]. Some differential models from population dynamics are given. The time dependent first-order hyperbolic partial differential equation (transport equation) is a general partial equation that describes the transport phenomena such as heat transfer, mass transfer, momentum transfer, etc. Some transport equations have multiple point-wise delay and advance which are classified as hyperbolic partial differential difference equations. These equations provide a tool to simulate several realistic physical and biological phenomena. For instance, in Stein's model [8], the distribution of neuronal firing intervals satisfies a transport equation of mixed type with appropriate initial-boundary condition given by

$$\frac{\partial F}{\partial t}(\upsilon,t)-(\upsilon/\tau_0)\frac{\partial F}{\partial \upsilon}(\upsilon,t)=pe[F(\upsilon-1,t)-F(\upsilon,t)] +pi[F(\upsilon+\upsilon_0,t)-F(\upsilon,t)], \quad (1)$$

$$F(\upsilon,0)=F_0(\upsilon),$$

where V_t equal the depolarization at time t; $F(\upsilon,t)$ equal the probability that $V_t \le \upsilon$ at time t F_0 is initial data. Here, it assume that excitatory and inhibitory impulse occur randomly with a frequency p_e/sec and p_i/sec, respectively. After each neuronal firing there is a refractory period of duration t_0, during which the impulse have no effect and the membrane depolarization V_t is reset to be zero. At times $t > t_0$, an excitatory impulse produces unit depolarization while inhibitory impulse produces υ_0 unit repolarization, and if the depolarization reaches a threshold of r units, the neuron fires. For sub- threshold levels, the depolarization decays exponentially between impulses with the time constant τ_0. To study the neuron variability in quantitative terms, Stein transformed this equation and obtain the characteristic function of the distribution and analyzed the mean and variance of the distribution. We refer to [8] for more detailed information about the assumptions of the model. We are interested to find the value of unknown F.

2. Formulation of the problem

The equation (1) given in the previous section is a first-order partial differential-difference equation. Motivated from the model described in the previous section, we consider the following general transport equation with point-wise delay and advance:

$$\frac{\partial u}{\partial t}(x,t)+a(x,t)\frac{\partial u}{\partial x}(x,t)=b(x,t)[u(x-\alpha,t)-u(x,t)] +c(x,t)[u(x+\beta,t)-u(x,t)] \quad (2)$$

with initial condition

$$u(x,0)=u_0(x) \quad (3)$$

where $a(x,t), b(x,t)$ and $c(x,t)$ are sufficiently smooth functions of x and t. α and β are small positive constants.

Let us suppose the region in which we want to find the solution of equation (2) is $0 \le x \le X$. Since the partial differential equation (2) is first order hyperbolic type with difference terms, according to the direction of characteristics, we require only one boundary condition. If $a > 0$, we need a boundary condition at left side of the domain i.e. at $x = 0$ and if $a < 0$, we need a boundary condition at right side of the domain i.e. at $x = X$, Morton [6]. Therefore the boundary interval conditions for this equation are given by

$$u(s,t)=\phi_1(s,t), \quad \forall s \in [-\alpha,0]; \quad \text{for } a>0,$$
$$u(s,t)=\phi_2(s,t), \quad \forall s \in [X, X+\beta]; \quad \text{for } a<0, \quad (4)$$

The Taylor series approximation for the delay and advance term in equation (2), yields

$$u(x-\alpha,t)=u(x,t)-\alpha u_x(x,t)+R_1$$
$$u(x+\beta,t)=u(x,t)+\beta u_x(x,t)+R_2$$

where R_1 and R_2 are the remainders in the Taylor series expansions s.t.,

$$R_1=\frac{1}{2!}\alpha^2\frac{\partial^2}{\partial x^2}u(\xi,t) \qquad x-\alpha<\xi<x$$

and

$$R_2=\frac{1}{2!}\beta^2\frac{\partial^2}{\partial x^2}u(\eta,t) \qquad x<\eta<x+\beta$$

By substituting the expansions $u(x-\alpha,t)$ and $u(x+\beta,t)$ values in (2), we get

$$u_t(x,t)+a(x,t)u_x(x,t)=b(x,t)[u(x,t)-\alpha u_x(x,t)+R_1-u(x,t)]$$
$$+c(x,t)[u(x,t)+\beta u_x(x,t)+R_2-u(x,t)]$$

For sufficiently small values of point-wise delay and advance, remainder terms R_1 and R_2 are negligible. Therefore

$$u_t+[a(x,t)+\alpha b(x,t)-\beta c(x,t)]u_x\approx 0 \qquad (5)$$

with initial and boundary-interval conditions,

$$u(x,0)=u_0(x),$$
$$u(0,t)=\phi_1(0,t), \quad \text{for} \quad (a+\alpha b-\beta c)>0 \qquad (6)$$
$$u(X,t)=\phi_2(X,t), \quad \text{for} \quad (a+\alpha b-\beta c)<0$$

In the following sections, a numerical method based on finite difference is developed to solve such type of initial and boundary value problems. The proposed method is analyzed for stability and convergence. Some test examples are given to validate convergence and computational efficiency of the proposed numerical algorithm.

3. Numerical Approximation

In this section, we construct numerical scheme based on the finite difference method [6]. We discuss first and second order explicit numerical approximations for the given equation (2) based on Lax-Friedrichs finite difference approximations. The differential equation (2) is hyperbolic and first-order with difference terms. For space time approximations based on finite differences, the (x,t) plane is discretize by taking mesh width Δx and time step Δt, and defining the gird points (x_j,t_n) by

$$x_j=j\Delta x,\ \ j=0,1,\dots,J-1,J;\ \ t_n=n\Delta t,$$
$$n=0,1,2,\dots$$

Now we look for discrete solution u_j^n that approximate $u(x_j,t_n),\ \forall j,n$.

3.1 Construction of the Numerical Scheme

In this approximation, we approximate the time derivative by forward difference and space derivative by centered difference and then we replace U_j^n by the mean value between U_{j+1}^n and U_{j-1}^n for stability purpose. Numerical scheme is given by

$$\frac{U_j^{n+1}-\dfrac{U_{j+1}^n+U_{j-1}^n}{2}}{\Delta t}+A_j^n\frac{U_{j+1}^n-U_{j-1}^n}{2\Delta x}=0$$

solving for U_j^{n+1}, we get

$$U_j^{n+1}=\frac{1}{2}(U_{j+1}^n+U_{j-1}^n)-\frac{1}{2}A_j^n\frac{\Delta t}{\Delta x}(U_{j+1}^n-U_{j-1}^n),$$
$$\forall j=1,2,\dots,J-1 \qquad (7)$$

where $\lambda=A_j^n\dfrac{\Delta t}{\Delta x}$.

together with initial and boundary-interval conditions are given by

$$U_j^0=u_0(x_j), \qquad j=1,\dots,J-1,$$
$$U_0^n=\phi_1(0,t_n), \qquad n=1,2,\dots$$
$$U_J^n=\phi_2(X,t_n), \qquad n=1,2,\dots$$

3.2 Stability Analysis

Definition: The finite difference method is called stable in the certain norm $\|.\|$ if there exists constant $C>0$, independent of the space step and time step such that

$$\|U^n\|\le C\|U^0\|, \qquad \forall\ n=1,2,\dots$$

now consider the finite difference scheme as given equation (7) i.e.

$$U_j^{n+1}=\frac{1}{2}\left(1-A_j^n\frac{\Delta t}{\Delta x}\right)U_{j+1}^n+\frac{1}{2}\left(1+A_j^n\frac{\Delta t}{\Delta x}\right)U_{j-1}^n$$

$$|U_j^{n+1}|\le\frac{1}{2}\left|\left(1-A_j^n\frac{\Delta t}{\Delta x}\right)\right||U_{j+1}^n|+\frac{1}{2}\left|\left(1+A_j^n\frac{\Delta t}{\Delta x}\right)\right||U_{j-1}^n|$$

taking the norm, we get

$$\|U^{n+1}\|_{L^\infty}=\sup_j|U_j^{n+1}|$$
$$\le\frac{1}{2}\left|\left(1-A_j^n\frac{\Delta t}{\Delta x}\right)\right|\sup_j|U_{j+1}^n|+\frac{1}{2}\left|\left(1+A_j^n\frac{\Delta t}{\Delta x}\right)\right|\sup_j|U_{j-1}^n|$$
$$\le\frac{1}{2}\left|\left(1-A_j^n\frac{\Delta t}{\Delta x}\right)\right|\|U^n\|_{L^\infty}+\frac{1}{2}\left|\left(1+A_j^n\frac{\Delta t}{\Delta x}\right)\right|\|U^n\|_{L^\infty}$$

If $\left|A_j^n\dfrac{\Delta t}{\Delta x}\right|\le 1$, the above inequality reduces to

$$\|U^{n+1}\|_{L^\infty}\le\|U^n\|_{L^\infty}$$

which implies the stability of the numerical scheme provided $\left|A_j^n\dfrac{\Delta t}{\Delta x}\right|\le 1$.

3.3 Error Analysis

The local truncation error of the numerical scheme is obtained by replacing the approximate solution U_j^n by exact solution $u(x_j,t_n)$ in the numerical scheme. If u is sufficiently smooth, the truncation error T_j^n of this finite difference scheme is given by

$$T_j^n = \frac{u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2}}{\Delta t} + A_j^n \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$\approx \left[u_t + \frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx} + O(\Delta t^2) + O(\frac{\Delta x^4}{\Delta t}) \right]_j^n$$

$$+ \left[A(u_x + \frac{1}{6}\Delta x^2 u_{xxx}) + O(\Delta x^4) \right]_j^n$$

$$= [u_t + Au_x]_j^n + \left[\frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx} + \frac{\Delta x^2}{6} A u_{xxx} \right]_j^n$$

$$+ O(\Delta x^4 + \Delta t^{-1}\Delta x^4 + \Delta t^2).$$

Since u is exact solution, we get

$$[u_t + Au_x]_j^n = 0$$

hence

$$T_j^n = \left[\frac{1}{2}\Delta t u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx} + \frac{\Delta x^2}{6} A u_{xxx} \right]_j^n$$

$$+ O(\Delta x^4 + \Delta t^{-1}\Delta x^4 + \Delta t^2).$$

which shows that the numerical scheme is consistent of order 2 in space and of order 1 in time as long as $\Delta t^{-1}\Delta x^2 \to 0$.

3.4 Convergence of the scheme

Definition: A finite difference scheme (Numerical method) is said to be convergent if for any fixed point (x^*, t^*) in a given domain $(0, X) \times (0, t_n)$,

$$x_j \to x^*, \ t_n \to t^* \quad \Rightarrow \quad U_j^n \to u(x^*, t^*)$$

the error in the approximation is given by

$$e_j^n = U_j^n - u(x_j, t_n).$$

Now U_j^n satisfies the finite difference scheme (7) exactly, while $u(x_j, t_n)$ leaves the remainder $T_j^n \Delta t$. Therefore the error is given by

$$e_j^{n+1} = \frac{1}{2}\left(1 - A_j^n \frac{\Delta t}{\Delta x}\right) e_{j+1}^n + \frac{1}{2}\left(1 + A_j^n \frac{\Delta t}{\Delta x}\right) e_{j-1}^n - \Delta t T_j^n$$

and $e_0^n = 0$.

Let $E^n = \max\left\{ \left| e_j^n \right|, \ j = 0, 1, \ldots, J \right\}$

Hence for $\left| A_j^n \frac{\Delta t}{\Delta x} \right| \le 1$,

$$E^{n+1} = \max_j \left| e_j^{n+1} \right| \le E^n + \Delta t \max_j \left| T_j^n \right| \quad \text{and } E^0 = 0$$

If we suppose that the truncation error is bounded i.e. $\left| T_j^n \right| \le T_{\max}$, then by induction method

$$E^n \le + n\Delta t T_{\max} \le t_n T_{\max},$$

which shows that the method has first-order convergent provided that the solution has bounded derivatives up to second order.

4. Numerical Experiments

In this section, we present some numerical examples to validate the predicted results established in the paper. We perform numerical computations using MATLAB. The maximum absolute errors for the considered examples are calculated using half mesh principle as the exact solution for the considered examples are not available [3]. We calculate the errors by refining the grid points. The error in the numerical approximation is given by

$$E(\Delta x, \Delta t) = \max_{0 \le j \le J, 0 \le n \le Nt} \left| U_{\Delta x}^{\Delta t}(j, n) - U_{\Delta x/2}^{\Delta t/2}(2j, 2n) \right|$$

In the following examples the domain of consideration is $\Omega = [0,1] \times [0, 0.7]$.

Example1. Consider the problem (2) with the following coefficients and initial- boundary conditions:

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}; \ b(x,t) = 5; \ \ c(x,t) = 10;$$

$$u(x,0) = \exp[-10(4x-1)^2]; \quad . \ u(s,t) = 0, \quad \forall s \in [-\alpha, 0].$$

Example2. Consider the problem (2) with the following coefficients and initial- boundary conditions:

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}; \quad b(x,t) = 1 + 2x^2t^2 + x^4;$$

$$c(x,t) = 1 + xt;$$

$$u(x,0) = \exp[-10(4x-1)^2]; \quad . \ u(s,t) = 0, \quad \forall s \in [-\alpha, 0] \ .$$

Example3. Consider the problem (2) with the following coefficients and initial- boundary conditions:

$$a(x,t) = \frac{1+x^2}{1+2xt+2x^2+x^4}; \quad b(x,t) = \frac{1}{1+2x^2t^2+x^4};$$

$$c(x,t) = \frac{1}{1+xt};$$

$$u(x,0) = \exp[-10(4x-1)^2]; \quad . \ u(s,t) = 0, \quad \forall s \in [-\alpha, 0]$$

Table 1: The maximum absolute error for example 1

$\Delta t \downarrow \ \Delta x \rightarrow$	1/100	1/200	1/400	1/800
$\Delta x/2$	0.0588	0.0296	0.0149	0.0071
$\Delta x/4$	0.0294	0.0146	0.0069	0.0035
$\Delta x/8$	0.0145	0.0068	0.0034	0.0017
$\Delta x/16$	0.0067	0.0033	0.0016	0.0008

Table 2: The maximum absolute error for example 2

$\Delta t \downarrow \ \Delta x \rightarrow$	1/100	1/200	1/400	1/800
$\Delta x/2$	0.0582	0.0294	0.0147	0.0071
$\Delta x/4$	0.0292	0.0144	0.0068	0.0034
$\Delta x/8$	0.0141	0.0067	0.0033	0.0016
$\Delta x/16$	0.0065	0.0031	0.0015	0.0007

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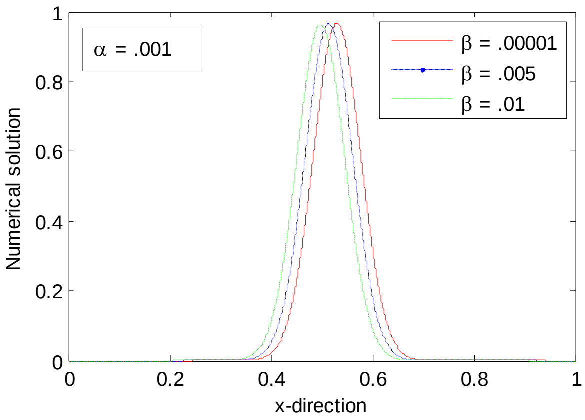

Figure1: The numerical solution for Example 1 at $t = 0.5$.

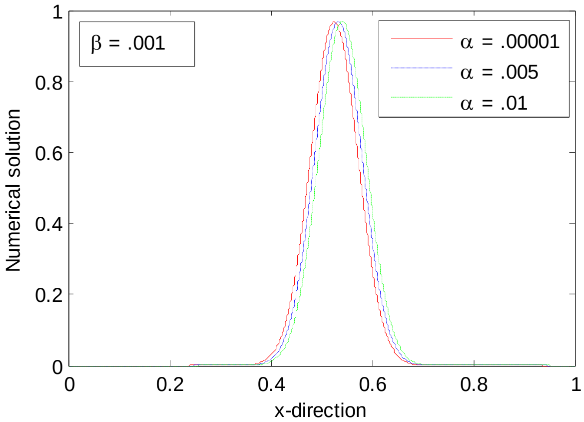

Figure2: The numerical solution for Example 1 at $t = 0.5$.

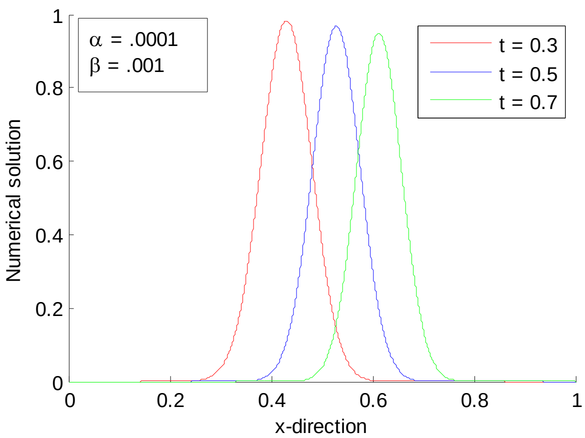

Figure3: The numerical solution of Example 2 for different values of *t*.

Table 3: The maximum absolute error for example 3

$\Delta t \downarrow \Delta x \rightarrow$	1/100	1/200	1/400	1/800
$\Delta x/2$	0.0580	0.0289	0.0146	0.0069
$\Delta x/4$	0.0291	0.0143	0.0068	0.0033
$\Delta x/8$	0.0145	0.0067	0.0032	0.0015
$\Delta x/16$	0.0068	0.0033	0.0016	0.0007

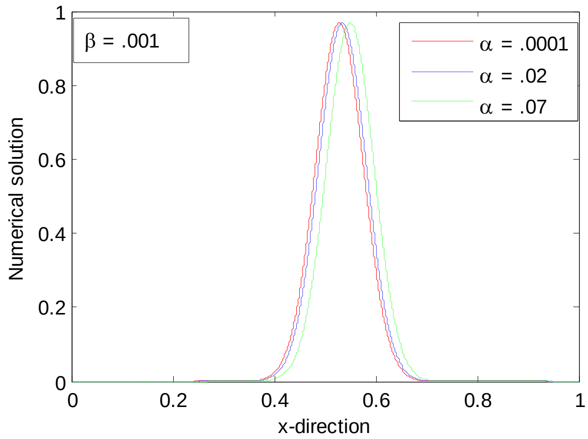

Figure4: The numerical solution for Example 3 at $t = 0.5$.

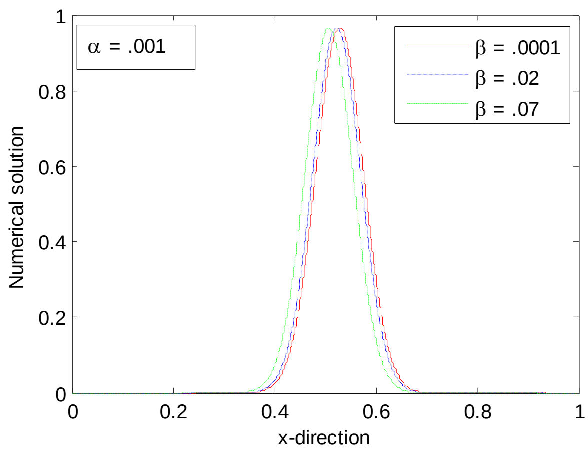

Figure5: The numerical solution for Example 3 at $t = 0.5$.

5. Conclusion

In this paper, a first-order hyperbolic partial differential difference equation for the distribution of neuronal firing based on the Stein's Model [8]. For finding the numerical solution of the initial and boundary value problem, a numerical scheme based on upwind finite difference is developed. The maximum absolute error are computed and tabulated in tables 1-3 for the considered examples with $\alpha = 0.001$ and $\beta = 0.001$. The error table illustrates that the method is first order convergent in temporal and second order spatial directions. Basically in this paper, we compare the results as already discussed by Sharma and Singh [7]. Our results are better due to second order convergence of scheme. The graphs of the solution of the considered examples for different values of point-wise delay and advance are plotted in Figures 1-5 to examine the effect of point-wise delay as well as advance on the solution behavior of the problem. We observe that if we fix α and increase the value of β, impulse moves towards left see (fig.1 and 5) while fixing β and increase the value of α, impulse moves towards right see (fig.2 and 4). Now fixing both α and β, the impulse moves towards right with the time see (fig.3).

References

[1] Causon, D. M., and Mingham, C.G., "Introductory finite difference methods for PDEs", *Venturs Publishing ApS,* UK (2010).
[2] Eccles, J. C., "The Physiology of Synapses", *Springer*, Berlin, (1964).
[3] Doolan, E. P., Miller, J. J. H. and Schilders, W. H. A., "Uniform Numerical Methods for Problems with Initial and Boundary Layers", *Boole Press*, Dublin (1980).
[4] Hale, J. K., and Verduyn Lunel, S. M., "Introduction to Functional Differential Equations", *Springer-Verlag,* New York, (1993).
[5] Katz, B., "Depolarization of Sensory Terminals and the Initiation of Impulses in the Muscle Spindle", *J. Physiol*., **111** (1950), 261-282.
[6] Morton, K.W. and Mayers, D.F., "Numerical solution of partial differential equation", *Cambridge Univ-Press*, Cambridge, (1994).
[7] Sharma, Kapil K. and Singh, Paramjeet, Hyperbolic Partial Differential Difference Equation in the Mathematical Modeling of Neuronal Firing and its Numerical Solution, *Appl. Math. Comput,* **201** (2008), 229-238.
[8] Stein, R.B., A theoretical analysis of neuronal variability, *Biophysical. J*., **5** (1965), 173-194.

Author Profile

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