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Edge Domination in Euler Totient Cayley Graph

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Abstract: ***Graph Theory has been realized as one of the most flourishing branches of modern Mathematics finding widest applications in all most all branches of Sciences, Social Sciences, Engineering, Computer Science, etc. Number Theory is one of the oldest branches of Mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern. Nathanson [1] paved the way for the emergence of a new class of graphs, namely Arithmetic Graphs by introducing the concepts of Number Theory. Using the number theoretic function, the Euler totient function, we have defined an Euler totient Cayley graph and in this paper we study the Edge domination in Euler totient Cayley graph. This paper is devoted for the study of minimum edge cover, edge covering number, minimal edge dominating set and edge domination number of Euler totient Cayley graph in two cases when n is even and when n is odd.***

Keywords: Euler Totient Cayley Graph, Edge cover, Minimum edge cover, Edge covering number, Edge domination, Minimal edge domination, Edge domination number.

1. Introduction

The concept of edge domination is introduced by Mitchell and Hedetniemi [2]. Some results on edge domination are given by Arumugam and Velammal [3]. Cockayne and Mynhardt [4] have introduced that edge subsets may also be embedded into sets of functions and an analogous concept of convexity could also be developed. Here we consider Euler totient Cayley graph $G(Z_n, \phi)$. We determine minimum edge cover, minimum edge dominating sets, edge covering number and edge domination number of $G(Z_n, \varphi)$.

2. Euler Totient Cayley Graph and its Properties

Definition 2.1: The **Euler totient Cayley** graph is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2,......,n-1\}$ and the edge set is E = {(x, y) / x – y ∈ S or y – x ∈ S} and is denoted by $G(Z_n, \phi)$, where S denote the set of all positive integers less than n and relatively prime to n. That is $S = \{r / 1 \le r < n \text{ and } GCD(r, n) = 1\}$, $|S| = \varphi(n)$.

Now we present some of the properties of Euler totient Cayley graphs studied by Madhavi [5].

1. The graph $G(Z_n, \varphi)$ is $\varphi(n)$ – regular and has $\frac{n\varphi(n)}{2}$ edges.
2. The graph $G(Z_n, \varphi)$ is Hamiltonian and hence it is connected.
3. The graph $G(Z_n, \varphi)$ is Eulerian for n ≥ 3.
4. The graph $G(Z_n, \varphi)$ is bipartite if n is even.
5. The graph $G(Z_n, \varphi)$ is complete if n is a prime.

3. Edge Domination

Definition 3.1: Let G(V, E) be a graph. A subset F of E is called an **edge cover** of G, if F contains all the vertices of G. A **minimum edge cover** is one with minimum cardinality.
The number of edges in a minimum edge cover of G is called the **edge covering number** of G and it is denoted by $\beta'(G)$.

Definition 3.2: Let G(V, E) be a graph. A subset F of E is called an **edge dominating set (EDS)** of G if each edge in E – F is adjacent to at least one edge in F.

An edge dominating set (EDS) F is called a **minimal edge dominating set (MEDS)** if no proper subset of F is an EDS of G. The minimum cardinality among all edge dominating sets of G is called an **edge domination number** of G and is denoted by $\gamma'(G)$.

Theorem 3.3: The minimum edge cover of $G(Z_n, \varphi)$, when n is odd is given by {(0, 1), (2, 3),…..,(n – 3, n – 2), (n – 1, 0)}.

Proof: Consider $G(Z_n, \varphi)$, where n is odd. Let E denote the edge set of $G(Z_n, \varphi)$.

Consider the set F of ordered pairs of vertices given by
F = {(0, 1), (2, 3),…..,(n – 3, n – 2), (n – 1, 0)}.
Since (2i + 1) – (2i) = 1 ∈ S, each ordered pair (2i, 2i + 1), $0 \le i \le \frac{n-1}{2}$ in F is an edge of $G(Z_n, \varphi)$, so that F becomes a subset of E. Also the edges in F contain all the vertices of $G(Z_n, \varphi)$.

Therefore F is an edge cover of $G(Z_n, \varphi)$. Now we show that F is minimum.
Let us consider the edge set F – { e_i } where e_i = (2i, 2i + 1) ∈ F for any i = 0, 1, 2,……, $\frac{n-1}{2}$.
Consider the vertices 2i and 2i + 1.

The vertex 2i is incident with the edge (2i – 1, 2i) and (2i, 2i + 1). But the edge (2i – 1, 2i) $\notin$ F and the edge (2i, 2i + 1) = { e_i } $\notin$ F – { e_i }.

Similarly (2i + 1) is incident with the edge (2i, 2i + 1) and (2i + 1, 2i + 2). But the edge (2i, 2i + 1) = { e_i } $\notin$ F – { e_i } and the edge (2i + 1, 2i + 2) $\notin$ F.

i.e., the set of edges of F – { e_i } does not contain all the vertices of G.

So F – { e_i } is not an edge cover of $G(Z_n, \varphi)$.

$\Rightarrow$ F is a minimum edge cover of G. ■

Theorem 3.4: The minimum edge cover of $G(Z_n, \varphi)$, when n is even is given by
{(0, 1) (2, 3),….,(n – 2, n – 1)}.

Proof: Consider $G(Z_n, \varphi)$, where n is even. Let E denote the edge set of $G(Z_n, \varphi)$

Consider the set F of ordered pairs of vertices given by
F = {(0, 1), (2, 3),……,(n – 2, n – 1)}.
Since (2i + 1) – (2i) = 1 $\in$ S, each ordered pair (2i, 2i + 1), 0 $\le$ i $\le \frac{n-2}{2}$ in F is an edge of G.

So F becomes a subset of E.

Also the edges in F contain all the vertices of $G(Z_n, \varphi)$.

$\Rightarrow$ F is an edge cover of G.
Now we check for the minimality of F.

Let us consider F – { e_i }, where e_i = (2i, 2i + 1) $\in$ F, for any i = 0, 1, 2…….., $\frac{n-1}{2}$.

Consider the vertices 2i and 2i + 1.
The vertex 2i is incident with the edges (2i – 1, 2i) and (2i, 2i + 1). But the edge (2i – 1, 2i) $\notin$ F and the edge (2i, 2i + 1) = { e_i } $\notin$ F – { e_i }.

Similar is the case with the vertex 2i+1.

So the edges of F – { e_i } does not cover all the vertices of G.

i.e., F – { e_i } is not an edge cover of G.

$\Rightarrow$ F is a minimum edge cover of $G(Z_n, \varphi)$. ■

Corollary 3.5: The edge covering number of $G(Z_n, \varphi)$ is $\frac{n}{2}$ when n is even and $\frac{n+1}{2}$ when n is odd.

Proof: If n is even then the n vertices of $G(Z_n, \varphi)$ can be paired into $\frac{n}{2}$ distinct pairs of vertices (2i, 2i + 1), 0 $\le$ i $\le \frac{n-2}{2}$, it follows that the cardinality of F = {(0, 1), (2, 3),….,(n – 2, n – 1)} is $\frac{n}{2}$. i.e., $\beta'(G(Z_n, \varphi)) = \frac{n}{2}$.

If n is odd then the n vertices of $G(Z_n, \varphi)$ can be paired into $\frac{n+1}{2}$ distinct pairs of vertices (2i, 2i + 1), 0 $\le$ i $\le \frac{n-1}{2}$.

Hence the cardinality of F = {(0, 1), (2, 3),….,(n – 1, 0)} is $\frac{n+1}{2}$.

i.e., $\beta'(G(Z_n, \varphi)) = \frac{n+1}{2}$. ■

Theorem 3.6: The set of edges {(1, 2), (3, 4),….,(n – 2, n – 1)} forms a minimal edge dominating set of $G(Z_n, \varphi)$, when n is odd.

Proof: Consider $G(Z_n, \varphi)$, when n is odd. Let E denote the edge set of $G(Z_n, \varphi)$.

Let F = {(1, 2), (3, 4),…..,(n – 2, n – 1)}.
Since 2i – (2i – 1) = 1$\in$S, each ordered pair (2i –1, 2i), 0 < i $\le \frac{n-1}{2}$ in F is an edge of $G(Z_n, \varphi)$.

$\Rightarrow$ F is a subset of E.
Let (s, t) $\in$ E – F, where s $\ge$ 0 and t $\neq$ s + 1.
Consider the edge (s, s + 1) in F, where s $\neq$ 2i, i = 1, 2…….. $\frac{n-1}{2}$.

Obviously this edge is adjacent with (s, t).
So every edge of E – F is adjacent with at least one edge of F.
$\Rightarrow$ F is an edge dominating set.
We now check for the minimality of F.

Delete an edge e = (1, 2) from F. Then there is an edge (0, 1) $\in$ E, such that it is not adjacent with any edge of F – { e }, because the edge (0, 1) is adjacent with the edges (1, 2) and (0, n – 1) for n > 1 as 1 $\in$ S. Also the edge (0, 1) is adjacent with the edges (0, q) and (1, r),

where 1 < q < (n – 1), 2 < r < n. Let $|q – 0|$ = k_1 and $|r – 1|$ = k_2, where k_1, k_2 > 1 $\in$ S.
But none of these edges belong to F – { e }, as the edges in F are of the form

(2i – 1, 2i) where i = 1, 2,…., $\frac{n-1}{2}$.

i.e., F – { e } is not an edge dominating set of $G(Z_n, \varphi)$.

Hence F is a minimum edge dominating set of $G(Z_n, \varphi)$. ■

Corollary 3.7: The edge domination number $\gamma'(G(Z_n, \varphi)) = \frac{n-1}{2}$, when n is odd.

Proof: By Theorem 3.6, the minimum edge dominating set of $G(Z_n,\varphi)$ is

F = {(1, 2), (3, 4), ……,(n – 2, n – 1)}.

Since the (n – 1) vertices can be paired into $\frac{n-1}{2}$ distinct pairs of vertices (2i – 1, 2i),

$0 < i \leq \frac{n-1}{2}$, it follows that the cardinality of F is $\frac{n-1}{2}$.

Therefore $\gamma'(G(Z_n,\varphi)) = \frac{n-1}{2}$. ■

Theorem 3.8: The set of edges {(0, 1), (2, 3),……,(n − 2, n − 1)} forms a minimal edge dominating set of $G(Z_n,\varphi)$, when n is even.

Proof: Consider $G(Z_n,\varphi)$, where n is even. Let E denote the edge set of $G(Z_n,\varphi)$.

Let F = {(0, 1), (2, 3),…,(n−2, n – 1)}.

Since (2i +1) – 2i = 1 ∈ S, each ordered pair (2i, 2i + 1), $0 \leq i \leq \frac{n-2}{2}$ in F is an edge of $G(Z_n,\varphi)$. i.e., F is a subset of E in $G(Z_n,\varphi)$.

Let (s, t) ∈ E – F , where s ≥ 0 and t ≠ s + 1.

Consider the edge (s, s + 1) in F, where s ≠ 2i, i = 0, 1, 2,…, $\frac{n-1}{2}$.

Obviously this edge is adjacent with (s, t).

Hence it follows that F is an edge dominating set of $G(Z_n,\varphi)$.

Now we prove that F is minimal. Delete an edge e = (0, 1) from F.

Then the edge (0, 1) belongs to E – F, which is adjacent with (1, 2) and (0, n − 1) for n > 1.

Also it is adjacent with the edges (0, q) and (1, r) where 1 < q < n − 1, 2 < r < n.

Let $|0 - q| = k_1$ and $|1 - r| = k_2$ where $k_1, k_2 > 1 \in S$.

But none of these edges belong to F – {e} as the edges in F are of the form

(2i, 2i + 1) where i = 1, 2,…., $\frac{n-2}{2}$.

i.e., F – {e} is not an edge dominating set.

Hence F is a minimum edge dominating set of $G(Z_n,\varphi)$.

Corollary 3.9: The edge domination number $\gamma'(G(Z_n,\varphi)) = \frac{n}{2}$, when n is even.

Proof: By Theorem 3.8, a minimum edge dominating set of $G(Z_n,\varphi)$ when n is even is given by F = {(0, 1), (2, 3),……….,(n – 2, n – 1)}.

The n vertices {0, 1, 2, 3,….,n – 1} can be paired into $\frac{n}{2}$ distinct pairs of vertices (2i, 2i + 1), $0 \leq i \leq \frac{n-2}{2}$.

Hence it follows that the cardinality of F is $\frac{n}{2}$.

i.e., $\gamma'(G(Z_n,\varphi)) = \frac{n}{2}$. ■

4. Future Scope

There are several other dominating parameters which can be applied on Euler Totient Cayley Graph. The theorems which are derived in this paper can be applied practically for solving graph theoritical problems.

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