A Steiner Problem in Petersen Graph Which is NP - Complete

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Abstract: ***In this paper, we will discuss complexity theory, a specific problem known as R-Restricted Steiner problem in Petersen graph is NP complete and also every full component of a Steiner tree contains almost 4 terminals.***

Keywords: Steiner minimum tree, NP-complete, satisfiability, petersengraph.

1. Introduction

A Steiner minimum tree in a graph with R-terminals are interior points. The Steiner tree problem in graph called for brevity ST, defined in decisional form as follows:

- an undirected graph $G=(V,E)$
- a subset of the vertices $R\subseteq V$, called terminal nodes.
- a number $K\in N$.

There is a subtree of G that includes all the vertices of R. (ie. a spanning tree for R) and that contains atmost K edges.

Steiner tree problem has many applications especially when we have to plan a connectivity structure among different terminal points. For example, when we want to find an optimal way to build roads and railways to connect, a set of cities or decide routing policies over the internet for multicast traffic, usually from a source to many destinations.

This is a very important concept in complexity theory. In this work, to prove that R-Restricted steiner problem in Petersen graph is NP-complete and also every full component of a steiner tree contains almost 4 terminals.

2. Preliminaries

Definition 2.1
A Steiner tree is a tree in a distance graph which spans a given subset of vertices (Steiner point with the minimum total distance on its edges).

Definition 2.2
Let a connected graph $G=(V,E)$ and a set $K\subseteq Y$ of terminals. Then the Steiner minimum tree for K in G that is Steiner tree τ for K such that

$$|E(T)| = \min \{E(T')/T' \text{is a steiner tree for } K \text{ in } G\}$$

In the Steiner minimum tree problem, the vertices are divided into two parts, terminals and non terminal vertices. The terminals are the given vertices which must be included in the solution.

Example 2.1

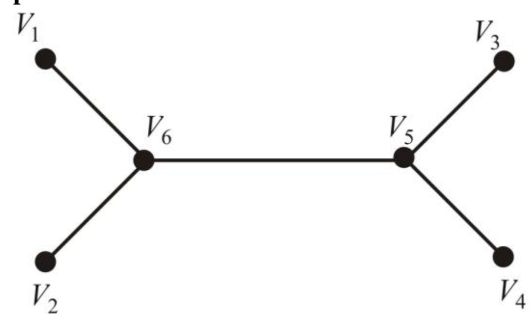

Figure 1: Steiner minimal tree

V_1, V_2, V_3, V_4 are terminals V_5 and V_6 is non-terminals.

Definition 2.3
A class of problems solvable by non deterministic polynomial time algorithm is called NP.

Definition 2.4
A problem is NP-complete if
1. It is an element of the class NP.
2. Another NP-complete problem is polynomial time reducible to it.

Definition 2.5
A Steiner minimum tree for K is given such that some of the terminals are interior points. Then we can decompose this tree (by splitting terminals).

Example 2.2
Steiner tree for three points.

Figure 2: Steiner Tree

[Note there is no direct connection between A, B and C].

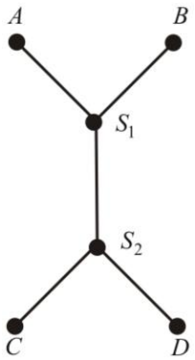

Figure 3: Steiner Tree

Solution for four points [Note that there are two Steiner points S_1 and S_2]

Definition 2.6

The Petersen graph is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

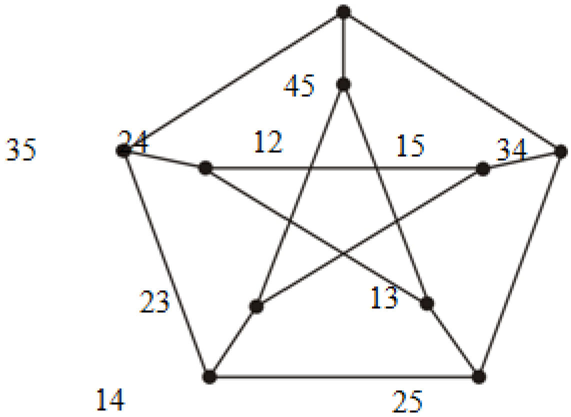

Figure 4: The Petersen graph

Example 2.3

The structure of the Petersen graph, using [5] = {1, 2, 3, 4, 5} as our 5-element set, we write the pair $\{a,b\}$ as ab or ba. Since 12 and 34 are disjoint, they are adjacent vertices when we form the graph, but 12 and 23 are not. For 2-set ab, there are 3 way to pick a 2 set from the remaining 3 elements of [5], so every vertex has degree 3.

The Petersen graph consists of 2 disjoint 5-cycles plus edges that pair up vertices on the two 5-cycles. The disjointness definition says that 12, 34, 25, 14, 35 in order to the vertices of a 5-cycle, and similarly this holds for the remaining vertices 13, 24, 15, 23, 45. Also 24 is adjacent to 35 and 15 is adjacent to 34 and so on.

Definition 2.7

A steiner minimum tree for *K* is given such that some of the terminals are interior points. Then we can decompose this tree into components so that terminals only occur as leaves of these components. Such a component is called full component. Figure 3 illustrates the decomposition of a Steiner tree into full components.

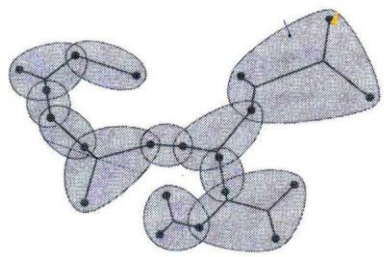

Figure 5: The full components of a Steiner Tree

3. Steiner Problem in Petersen Graph is NP-Complete

Result 1

R-Restricted Steiner problem in Petersen graph is NP-complete.

Proof

Let the Steiner problem in graph is ∈ NP, it is sufficient to show that R- Restricted Steiner problem in Petersen graph is infact NP-complete. Construct a Peterson graph of 10 vertices. We have drawn the Petersen graph in three ways.

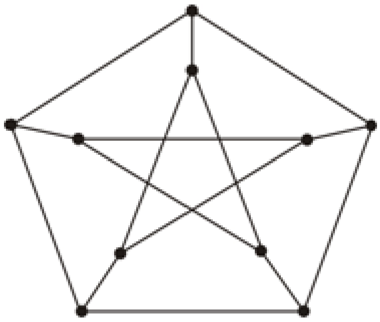

Figure 6: Petersen graph

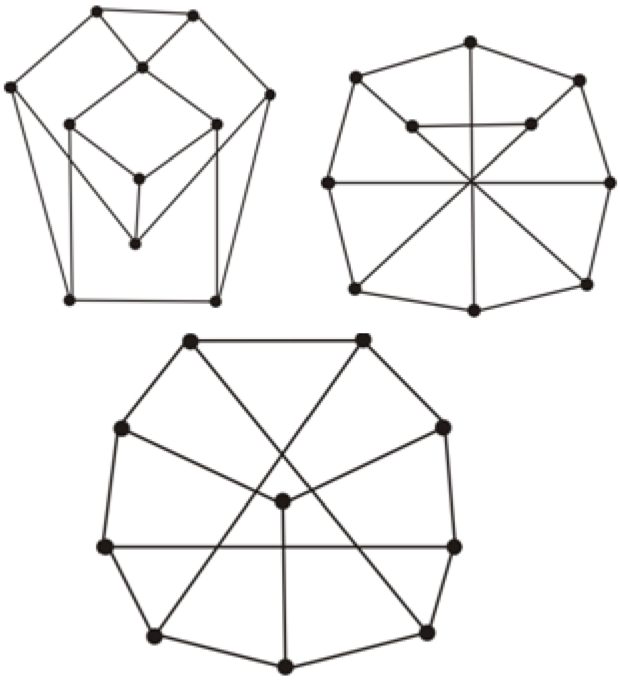

Figure 7: The three Avatars of Petersen graph

Petersen graph of 10 vertices and 15 Edges.

Now we take a Steiner tree from the Petersen graph of maximum number of 10 vertices in the following figure 8.

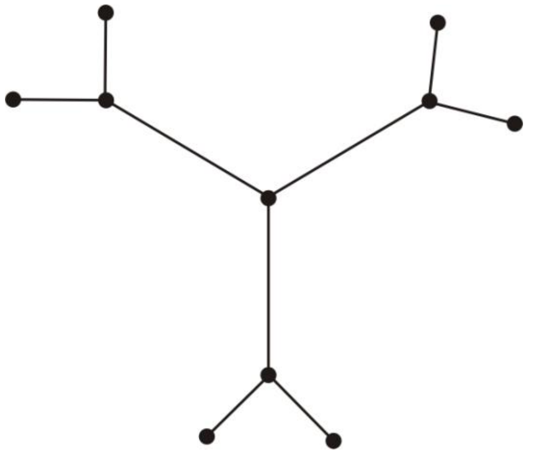

Figure 8: A Steiner tree in Petersen graph

We reduce 3 SAT to steiner problem in Petersen graphs. Let $x_1, x_2, \ldots, x_n$ be the variables. $c_1, c_2, \ldots, c_m$ the clauses in an arbitrary instance of 3SAT. Our aim is to construct a Petersen graph $G=(V,E)$ and a terminals set K, and a bound B such that Petersen graph contains steiner tree T for K at size at most B if and only if the given 3SAT instance is satisfiable.

Transforming 3SAT to Steiner problem in Peterson graph is constructed as follows. First we connect u and v by a variable path as shown in Figure 9.

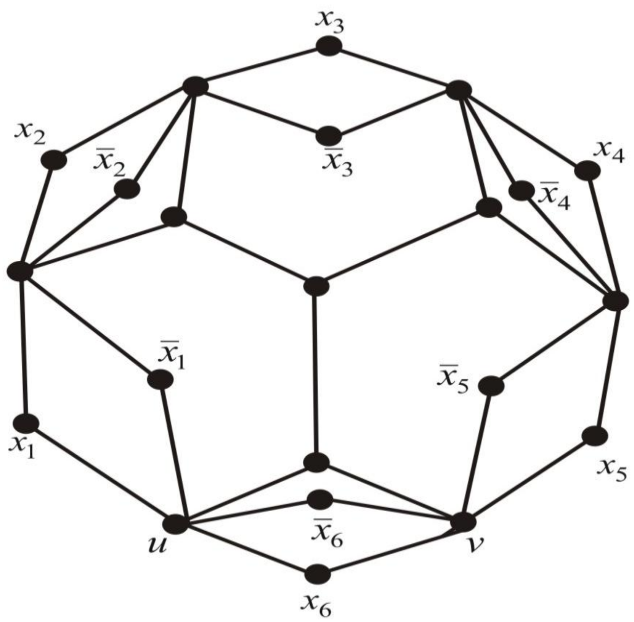

Figure 9: Transforming 3SAT to Steiner problem in Petersen graph

Then we create for every clause gadget consisting of a vertex C_i that is connected to the literals. contained in the clause C_i by paths of length $t = 2n+1$. As terminal set we choose $k = \{u, v\} \cup \{C_1 \ldots C_4\}$ and set $B = 2n + t.m.$

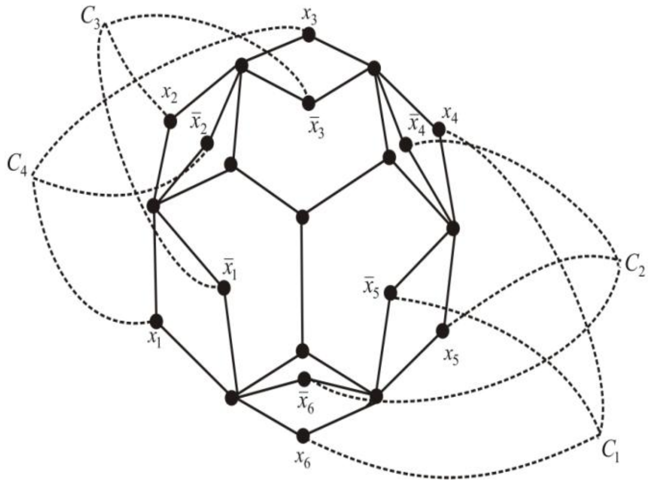

Figure 10: The clause gadget for the clause $C_1 = x_1 \vee \overline{x_2} \vee x_3.$

The dashed lines indicated paths of length $t = 2n+1$ from C_i to the appropriate vertices on the variable path. Assume first that 3SAT for instance is satisfiable let $x_i \in P$ if x_i is said to be true in this assignment, and $\bar{x}_i \in P$ otherwise.To construct a Steiner tree for K we start with $u-v$ path *P* is reflecting a satisfying assignment.

x_i is true for all variables. Hence we arrive from our SAT problem with six variables and 4 clauses for a 3SAT problem.

$C_1 = (x_6 \vee \bar{x}_5 \vee x_4)$ $C_2 = (\bar{x}_6 \vee x_5 \vee \bar{x}_4)$

$C_3 = (\bar{x}_1 \vee x_5 \vee \bar{x}_3)$ $C_4 = (x \vee \bar{x}_2 \vee x_3)$

With six variables and five clauses.

Now we take $n = 6$; to form the clauses $\{C_1, C_2, C_3, C_4\}$ and the terminal set $K = \{u, v\} \cup \{C_1, C_2, C_3, C_4\}$ and $B = 2n + t.m$

$$B = 2n + t.m$$

$$t = 2n+1$$

$$n = 6 \Rightarrow t = 2(6)+1 \quad = 13$$

$$m = 4 \Rightarrow B = 2n + t.m$$

$$= 2(6) + 13.4$$

$$= 12 + 52 = 64$$

To construct a Steiner tree for K we starting with a $u-v$ path P reflecting a satisfying assignment.

Next observe that for every clause the vertex C_i can be connected to P by path of Length t.

In this way we obtain a Steiner tree for K of Length $2n + t.m = B.$

On the other hand, we assume now that T is a Steiner tree for K of Length at most B, Trivially for each clause to the vertex C_i has to be connected to the variable path.

Then $|E(T)| \geq (m+1).t > B,$

$$\geq (4+1).13 > 64$$

$\geq 5.13 > 64$
$\geq 65 > 64 \Rightarrow$ contradiction

This show that u and v can only be connected along the variable path, which requires at least 2n edges.

In this graphs $u - v$ path contains 24 edges and that each clause gadget is connected to this path using exactly t edges.

In this graph each clause gadget is connected to this path using exactly 12 edges. Thus $u - v$ path reflects a satisfying assignment.

In the figure 3, the Steiner tree contains 6 terminals.

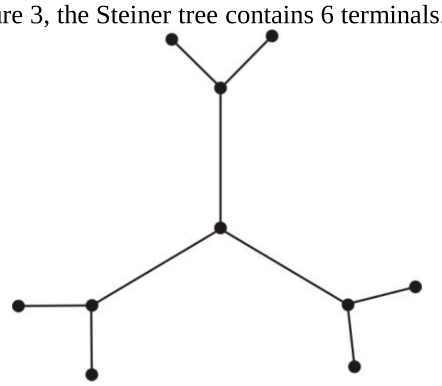

Figure 11: A Steiner minimum tree in Petersen graph

This implies that every full component of a Petersen graph contains at most four terminals.

Hence, Steiner tree in Petersen graph contains at most 4 terminals. This implies that R-Restricted Steiner Problem in Petersen graph is NP-Complete.

Result 1
Every $u - v$ Path of Steiner tree in Petersen graph is NP-Complete and every $u - v$ Path of Petersen graph contains exactly 2n edges.

Result 2
Transforming 3SAT to Steiner Problem in Petersen graph is NP-Complete.

Result 3
A Steiner tree of Petersen graph contains at most 4 terminals.

4. Conclusion

With Steiner tree in Petersen graph, we conclude that every $u - v$ Path contains exactly 2n edges, transforming 3SAT to Steiner problem in Petersen graph is NP-Complete and every full component of a Steiner tree in Petersen graph contains at most 4 terminals.

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