Extraction of Cantor Middle $(\omega = 2/5, 3/7)$ from Non-Reducible Farey Subsequence

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Abstract: ***From the paper of "Farey to Cantor" motivate to convert the Non reducible Farey Sequence of various order into various Cantor sequence of interval by remove some Non-Reducible Farey fractions.***

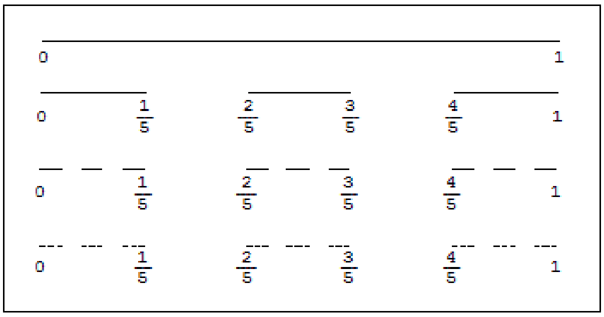
Keywords: Farey Sequence, Non-Reducible Farey Sequence, Cantor Middle $(\omega = 2/5, 3/7)$ Sequence, Non Reducible Farey N-Subsequence.

1. Introduction

The Farey Sequene is a pattern that has its origin in quite common numbers. The Farey fractions can be found in all sorts of different applications. The Farey sequence was so named for British born geologist, John Farey (1766-1826). In 1816 Farey wrote about the "curious nature of vulgar fractions" in the publication Philosophical Magazine. Given a sequence (F_N) are made up of fractions in lowest terms where the denominator is less than or equal a number N. When the fractions of F_1 are added together from the mediant property, $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$, a new fractions falls between the original two is generated. This fraction is called the mediant. The next series is found by adding the first two fractions of F_2 to find the mediant $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$. One finds the mediant of the last two fractions in F_2, $\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$, and the next Farey sequence is found. This procedure of finding the mediant between each pair of fraction in the previous Farey sequence is repeated to find the next sequence. In this paper we discuss only the extraction of Cantor Middle $(\omega = 2/5)$ set and Cantor Middle $(\omega = 3/7)$ set from Non-Reducible Farey Sequence. The Farey fractions lie in [0,1]. Similarly the Cantor Middle set lie in [0,1]. Here we try to construct the Cantor Middle set from Farey sequence

2. The Cantor Middle$-(\omega = 2/5)$ Set:

The interval $[0,1]$ divided it into five equal subintervals. Remove the open intervals $\left(\frac{1}{5}, \frac{2}{5}\right)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$ such that the Cantor set contains $\left[0, \frac{1}{5}\right] \cup \left[\frac{2}{5}, \frac{3}{5}\right] \cup \left[\frac{4}{5}, \frac{5}{5}\right]$ .Again subdivide each of these three remaining intervals into five equal subintervals and from each remove the second and fourth open subinterval. This process is continued to obtain a sequence of closed intervals.



2.1 Lemma

Construct the Cantor quintuple set $\mathbb{C}(5)$ from $\left[\frac{k}{5^{n-1}}, \frac{k+1}{5^{n-1}}\right]$, where n denotes the iteration.

Proof

Consider the closed interval $\left[\frac{k}{5^{n-1}}, \frac{k+1}{5^{n-1}}\right]$ for the initial iteration and multiply 5 with numerator and divide it into five equal subintervals and remove the open interval of middle second and fourth of five portions of the interval $\left[\frac{5k}{5^{n-1}}, \frac{5(k+1)}{5^{n-1}}\right]$ to obtain the Cantor set intervals

$$\mathbb{C}(5) = \left[\frac{5k}{5^{n-1}}, \frac{5(k+1)}{5^{n-1}}\right] - \left\{\left(\frac{5k}{5^{n-1}}, \frac{5k+2}{5^{n-1}}\right) \cup \left(\frac{5k+3}{5^{n-1}}, \frac{5k+4}{5^{n-1}}\right)\right\}$$

$$= \left[\frac{5k}{5^{n-1}}, \frac{5k+1}{5^{n-1}}\right] \cup \left[\frac{5k+2}{5^{n-1}}, \frac{5k+3}{5^{n-1}}\right] \cup \left[\frac{5k+4}{5^{n-1}}, \frac{5k+5}{5^{n-1}}\right]$$

From the next iteration, again we multiply the numerator of previous Cantor set interval and split into five sub interval to get current Cantor set intervals. This process is repeated until we get a Cantor set of required order.

3. Non-Reducible Farey Sequence

Consider the Non-Reducible Farey Sequence into intervals, in these intervals the fractions cannot be repeated. Writing the Non-Reduced Farey fractions as sequence, $\bar{F}_N = \left\{\frac{x_0}{N}, \frac{x_1}{N}, \frac{x_2}{N}, \frac{x_3}{N}, \ldots, \frac{x_{N-3}}{N}, \frac{x_{N-2}}{N}, \frac{x_{N-1}}{N}, \frac{x_N}{N}\right\}$, where $x_i = x_{i-1} + 1$, $x_0 = 0$, $x_N = N$ and $i = 0,1,\ldots N$. The fractions of the sequence $\bar{F}_N$ are partition into closed intervals $\left[\frac{x_0}{N}, \frac{x_1}{N}\right], \left[\frac{x_2}{N}, \frac{x_3}{N}\right], \ldots, \left[\frac{x_{N-3}}{N}, \frac{x_{N-2}}{N}\right], \left[\frac{x_{N-1}}{N}, \frac{x_N}{N}\right]$.

3.1 Farey sequence

The sequence of all reduced fractions with denominators not exceeding N listed in order of their size is called the Farey sequence of order N.

The Farey Sequence of order 5 is
$F_6 = \left\{\frac{0}{1},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{1}{1}\right\}$

3.2 Non Reducible Farey Sequence

The Sequence of non-reduced fractions with denominators not exceeding N listed in order of their size is called Non Reducible Farey Sequence of order N.

The Non Reducible Farey Sequence of order 6 is
$\widetilde{F_6} = \left\{\frac{0}{1}=\frac{0}{6},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{2}{6},\frac{2}{5},\frac{3}{6},\frac{3}{5},\frac{4}{6},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{6}{6}=\frac{1}{1}\right\}$

3.3Non Reducible Farey N-Subsequence

The sequence of non-reduced Farey fractions with denominators equal to the order of the size N is called Non Reducible Farey N-Subsequence.

The Non Reducible Farey N-Subsequence of order 6 is
$\widetilde{F_6} = \left\{\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6}\right\}$

3.4 Complement of a sequence

The numerator of each element of a sequence subtracted from its order is called the complement of a sequence.

3.5 Left Half Non-Reducible Farey N-Subsequence

The sequence $\left(\widetilde{F_N} < \frac{1}{2}\right)$ contains the non reducible Farey fractions whose denominators are equal to N and the fractions do not exceed $\frac{1}{2}$.

$$\left(\widetilde{F_N} < \frac{1}{2}\right) = \left\{\frac{h}{N} : h = 0,1,\dots,\left(\frac{N-1}{2}\right)\right\}$$

3.6 Right Half Non-Reducible Farey N-Subsequence

The sequence $\left(\widetilde{F_N} > \frac{1}{2}\right)$ is the complement of $\left(\widetilde{F_N} < \frac{1}{2}\right)$, The sequence $\left(\widetilde{F_N} > \frac{1}{2}\right)$ contains the Farey fractions(not reducible) whose denominators are equal to N.

$$\left(\widetilde{F_N} > \frac{1}{2}\right) = \left\{\frac{N-h}{N} : h = 0,1,\dots,\left(\frac{N-1}{2}\right)\right\}$$

3.7 Theorem

The Cantor quintuple set $C(5)$ is the set of numbers in $[0,1]$ which can be extracted from $\widetilde{F_{5^n}}$.

Proof

Consider the interval $\left[\frac{k}{5^{n-1}},\frac{k+1}{5^{n-1}}\right]$, where n denotes the iteration value and always $k = 0$.

Let $n = 1$,

$$\left[\frac{k}{5^{n-1}},\frac{k+1}{5^{n-1}}\right] = [0,1]$$

In both Farey and Cantor the partition start with $[0,1]$.

For $n = 2$, consider the interval $\left[\frac{k}{5^{n-1}},\frac{k+1}{5^{n-1}}\right]$ and multiply 5 with k and $k+1$ to get

$$\left[\frac{5k}{5^{n-1}},\frac{5(k+1)}{5^{n-1}}\right] = \left\{\left[\frac{5k}{5^{n-1}},\frac{5k+1}{5^{n-1}}\right] \cup \left[\frac{5k+2}{5^{n-1}},\frac{5k+3}{5^{n-1}}\right] \cup \left[\frac{5k+4}{5^{n-1}},\frac{5k+5}{5^{n-1}}\right]\right\}$$

In each interval the boundary points are non-reducible Farey fractions. Here, we remove $\left(\frac{5k+1}{5^{n-1}},\frac{5k+2}{5^{n-1}}\right)$ from $\left[\frac{5k}{5^{n-1}},\frac{5(k+1)}{5^{n-1}}\right]$ to get left half of Cantor sequence interval.

For $n = 3$,

$$\left[\frac{25k}{5^{n-1}},\frac{25(k+1)}{5^{n-1}}\right] = \left\{\left[\frac{25k}{5^{n-1}},\frac{25k+1}{5^{n-1}}\right] \cup \left[\frac{25k+2}{5^{n-1}},\frac{25k+3}{5^{n-1}}\right] \cup \dots \cup \left[\frac{25k+24}{5^{n-1}},\frac{25k+25}{5^{n-1}}\right]\right\}$$

From the left half of the Farey interval,namely,

$$\left[\frac{25k}{5^{n-1}},\frac{25k+1}{5^{n-1}}\right],\left[\frac{25k+2}{5^{n-1}},\frac{25k+3}{5^{n-1}}\right],\dots,\left[\frac{25k+12}{5^{n-1}},\frac{25k+13}{5^{n-1}}\right]$$

the interval

$$\mathcal{G}_{n-2} = \left(\frac{5^{n-2}(5k+1)}{N},\frac{5^{n-2}(5k+2)}{N}\right)$$

is removed successively .

For $n \geq 4$,

$$\left[\frac{5^{n-1}k}{5^{n-1}},\frac{5^{n-1}(k+1)}{5^{n-1}}\right] = \left\{\left[\frac{5^{n-1}k}{5^{n-1}},\frac{5^{n-1}k+1}{5^{n-1}}\right] \cup \left[\frac{5^{n-1}k+2}{5^{n-1}},\frac{5^{n-1}k+3}{5^{n-1}}\right] \cup \dots \cup \left[\frac{5^{n-1}k+(5^{n-1}-1)}{5^{n-1}},\frac{5^{n-1}k+5^{n-1}}{5^{n-1}}\right]\right\}$$

The left half Farey intervals are

$$\left(\widetilde{F_N} < \frac{1}{2}\right) = \left\{\left[\frac{5^{n-1}k}{5^{n-1}},\frac{5^{n-1}k+1}{5^{n-1}}\right],\left[\frac{5^{n-1}k+2}{5^{n-1}},\frac{5^{n-1}k+3}{5^{n-1}}\right],\left[\frac{5^{n-1}k+4}{5^{n-1}},\frac{5^{n-1}k+5}{5^{n-1}}\right],\dots,\left[\frac{5^{n-1}k+\left(\frac{5^{n-1}-1}{2}\right)}{5^{n-1}},\frac{5^{3}k+\left(\frac{5^{n-1}-1}{2}\right)+1}{5^{n-1}}\right]\right\}$$

The removable set of fractions is $\mathcal{G}_{n-2} \cup \mathfrak{D}_n$

Where $\mathfrak{D}_n =$

$$\left\{\left(\frac{5(2.5^{n-2}+1)}{N},\frac{5(2.5^{n-2}+2)}{N}\right),\left(\frac{5(2.5^{n-2}+3)}{N},\frac{5(2.5^{n-2}+4)}{N}\right),\dots,\left(\frac{5(2.5^{n-2}+2(5^{n-3}+5^{n-4}+\dots+5^{n-(n-1)})+1)}{N},\frac{(2.5^{n-2}+2(5^{n-3}+5^{n-4}+\dots+5^{n-(n-1)}+1))}{N}\right)\right\}$$

The set $\mathfrak{D}_n$ may in any one of the following forms

1) The numerator of the lower bound of the interval is of the form $5^{\alpha}, \alpha$ is an integer and the upper bound is twice the numerator of lower bound.
2) The numerator of the lower bound of the interval is a scalar smultiple of 5, then add five with the numerator of the lower bound to get the upper bound.
3) The numerator of the lower bound of the interval is of the form $5^{\alpha} + l$, α is an integer and l is any one of the integers $1,2,3,4$ then the upper bound is $5^{\alpha} + l + \alpha$.

$$\left(C_N < \frac{1}{2}\right) = \left\{\left(\widetilde{F_N} < \frac{1}{2}\right) / \{\mathcal{G}_{n-2}\} \cup \{\mathfrak{D}_n\}\right\}$$

In the n^{th} iteration the removable fractions are identified from the removable fractions of $(n-1)^{th}$ iteration and also using the set $\mathcal{G}_{n-2}$ and $\mathfrak{D}_n$ in left half of non-reduced Farey sub interval to get left half Cantor set interval and then determine the complement of the left half Cantor set fractions to get all the Cantor set interval.

4. The Cantor Middle$-^{(\omega = 3/7)}$ SET

In Cantor middle-$3/7$ set remove the middle second, fourth and sixth of seven portions of the unit interval $[0,1]$. For the first iteration, the partitioned intervals are $\left[\frac{0}{7},\frac{1}{7}\right] \cup \left[\frac{2}{7},\frac{3}{7}\right] \cup \left[\frac{4}{7},\frac{5}{7}\right] \cup \left[\frac{6}{7},\frac{7}{7}\right]$. In the second iteration, the previous iteration intervals are again partitioned. The partitioned intervals of second iterations are

$$\left[\frac{0}{49},\frac{1}{49}\right] \cup \left[\frac{2}{49},\frac{3}{49}\right] \cup \left[\frac{4}{49},\frac{5}{49}\right] \cup \left[\frac{6}{49},\frac{7}{49}\right] \cup \left[\frac{14}{49},\frac{15}{49}\right] \cup \left[\frac{16}{49},\frac{17}{49}\right] \cup \left[\frac{18}{49},\frac{19}{49}\right] \cup \left[\frac{20}{49},\frac{21}{49}\right] \cup \left[\frac{28}{49},\frac{29}{49}\right] \cup \left[\frac{30}{49},\frac{31}{49}\right] \cup \left[\frac{32}{49},\frac{33}{49}\right] \cup \left[\frac{34}{49},\frac{35}{49}\right] \cup \left[\frac{42}{49},\frac{43}{49}\right] \cup \left[\frac{44}{49},\frac{45}{49}\right] \cup \left[\frac{46}{49},\frac{47}{49}\right] \cup \left[\frac{48}{49},\frac{49}{49}\right]$$

Repeating this process, we get the Cantor middle $3/7$ set.

4.1 Theorem:

Extraction of Cantor middle $\frac{3}{7}$ set from Farey sequence.

Proof:

Consider the interval $\left[\frac{k}{7^{n-1}},\frac{k+1}{7^{n-1}}\right]$, where n denotes the iteration.

Let $n = 1$,

$$\left[\frac{7^{n-1}k}{7^{n-1}},\frac{7^{n-1}(k+1)}{7^{n-1}}\right] = [0,1]$$

In this iteration, Farey and Cantor are same intervals.

For $n = 2$, the interval $\left[\frac{7^{n-1}k}{7^{n-1}},\frac{7^{n-1}(k+1)}{7^{n-1}}\right]$ can be written as

$$\left[\frac{7k}{7},\frac{7(k+1)}{7}\right] = \left\{\left[\frac{7k}{7},\frac{7k+1}{7}\right] \cup \left[\frac{7k+2}{7},\frac{7k+3}{7}\right] \cup \left[\frac{7k+4}{7},\frac{7k+5}{7}\right] \cup \left[\frac{7k+6}{7},\frac{7k+7}{7}\right]\right\}$$

The boundary points are non-reducible Farey fractions. The intervals $\left(\frac{7k+1}{7},\frac{7k+2}{7}\right),\left(\frac{7k+3}{7},\frac{7k+4}{7}\right),\left(\frac{7k+5}{7},\frac{7k+6}{7}\right)$ are removed from $\left[\frac{7k}{7},\frac{7(k+1)}{7}\right]$ to get Cantor set interval.

For $n \geq 3$, the successive removable fractions are identified from the set

$$\varphi_n = \left\{\left(\frac{7^{n-1}}{7^n},\frac{2.7^{n-1}}{7^n}\right),\left(\frac{3.7^{n-1}}{7^n},\frac{4.7^{n-1}}{7^n}\right),\left(\frac{5.7^{n-1}}{7^n},\frac{6.7^{n-1}}{7^n}\right)\right\}$$

Each closed interval in the $(n-1)$th iteration is multiplied and divided by 7 and written as union of four closed intervals as below. The partitioned intervals for $n = 3$ is given below for illustration

$$\left[\frac{7^2k}{7^{n-1}},\frac{7^2k+7}{7^{n-1}}\right] = \left\{\left[\frac{7^2k}{7^{n-1}},\frac{7^2k+1}{7^{n-1}}\right] \cup \left[\frac{7^2k+2}{7^{n-1}},\frac{7^2k+3}{7^{n-1}}\right] \cup \left[\frac{7^2k+4}{7^{n-1}},\frac{7^2k+5}{7^{n-1}}\right] \cup \left[\frac{7^2k+6}{7^{n-1}},\frac{7^2k+7}{7^{n-1}}\right]\right\}$$

$$\left[\frac{7^2k+14}{7^{n-1}},\frac{7^2k+21}{7^{n-1}}\right] = \left\{\left[\frac{7^2k+14}{7^{n-1}},\frac{7^2k+15}{7^{n-1}}\right] \cup \left[\frac{7^2k+16}{7^{n-1}},\frac{7^2k+17}{7^{n-1}}\right] \cup \left[\frac{7^2k+18}{7^{n-1}},\frac{7^2k+19}{7^{n-1}}\right] \cup \left[\frac{7^2k+20}{7^{n-1}},\frac{7^2k+21}{7^{n-1}}\right]\right\}$$

$$\left[\frac{7^2k+28}{7^{n-1}},\frac{7^2k+35}{7^{n-1}}\right] = \left\{\left[\frac{7^2k+28}{7^{n-1}},\frac{7^2k+29}{7^{n-1}}\right] \cup \left[\frac{7^2k+30}{7^{n-1}},\frac{7^2k+31}{7^{n-1}}\right] \cup \left[\frac{7^2k+32}{7^{n-1}},\frac{7^2k+33}{7^{n-1}}\right] \cup \left[\frac{7^2k+34}{7^{n-1}},\frac{7^2k+35}{7^{n-1}}\right]\right\}$$

$$\left[\frac{7^2k+42}{7^{n-1}},\frac{7^2k+49}{7^{n-1}}\right] = \left\{\left[\frac{7^2k+42}{7^{n-1}},\frac{7^2k+43}{7^{n-1}}\right] \cup \left[\frac{7^2k+44}{7^{n-1}},\frac{7^2k+45}{7^{n-1}}\right] \cup \left[\frac{7^2k+46}{7^{n-1}},\frac{7^2k+47}{7^{n-1}}\right] \cup \left[\frac{7^2k+48}{7^{n-1}},\frac{7^2k+49}{7^{n-1}}\right]\right\}$$

The remaining removable sets of intervals ψ_n, $(n = 3)$ are depicted as follows.

$$\left(\frac{7^2k+1}{7^{n-1}},\frac{7^2k+2}{7^{n-1}}\right),\left(\frac{7^2k+3}{7^{n-1}},\frac{7^2k+4}{7^{n-1}}\right),\left(\frac{7^2k+5}{7^{n-1}},\frac{7^2k+6}{7^{n-1}}\right),$$
$$\left(\frac{7^2k+15}{7^{n-1}},\frac{7^2k+16}{7^{n-1}}\right),\left(\frac{7^2k+17}{7^{n-1}},\frac{7^2k+18}{7^{n-1}}\right),\left(\frac{7^2k+19}{7^{n-1}},\frac{7^2k+20}{7^{n-1}}\right),$$
$$\left(\frac{7^2k+29}{7^{n-1}},\frac{7^2k+30}{7^{n-1}}\right),\left(\frac{7^2k+31}{7^{n-1}},\frac{7^2k+32}{7^{n-1}}\right),\left(\frac{7^2k+33}{7^{n-1}},\frac{7^2k+34}{7^{n-1}}\right),$$
$$\left(\frac{7^2k+43}{7^{n-1}},\frac{7^2k+44}{7^{n-1}}\right),\left(\frac{7^2k+45}{7^{n-1}},\frac{7^2k+46}{7^{n-1}}\right),\left(\frac{7^2k+47}{7^{n-1}},\frac{7^2k+48}{7^{n-1}}\right).$$

The removal of $\varphi_n \cup \psi_n$ from $\widetilde{F_N}$ will result in Cantor Middle $\frac{3}{7}$ set

5. Conclusion

It is observed that the identifications of removable sets for Cantor middle $\frac{2}{5}$, $\frac{3}{7}$ sets from $\widetilde{F_{5^n}}$ and $\widetilde{F_{7^n}}$ differs slightly in approach the other Cantor middle sets from Farey fractions can be studied and if possible we generalized.

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