

# A Steiner Problem in Coxeter Graph Which is NP – Complete

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**Abstract:** Our objects in this work is to obtain complexity theory, a specific problem known as Steiner problem in Coxeter graph which is NP complete and also it is worth mentioning that our result every full component of a Steiner tree contains almost 4 terminals involved therein.

**Keywords:** Steiner tree, satisfiability, NP-complete, Coxeter graph

## 1. Introduction

A Steiner minimum tree in a graph with  $R$ -terminals is interior points. The Steiner tree (ST) problem in graph called for brevity ST, defined in decisional form as follows:

- An undirected graph  $G = (V, E)$
- A subset of the vertices  $R \subseteq V$ , called terminal nodes.
- A number  $K \in \mathbb{N}$ .

There is a subtree of  $G$  that includes all the vertices of  $R$ . (i.e., a spanning tree for  $R$ ) and that contains at most  $K$  edges.

Steiner tree problem has many applications especially when we have to plan a connectivity structure among different terminal points. For example, when we want to find an optimal way to build roads and railways to connect, a set of cities or decide routing policies over the internet for multicast traffic, usually from a source to many destinations.

For many decision problems no polynomial time algorithm is known. Nevertheless some of these problems have a property which is not inherent to every decision problem, there exists algorithm which, if presented with an instance of the problem [i.e., a graph  $G$  with terminal set  $K$ , and a bound  $B$ , respectively a Boolean Formula  $F$ ]. These algorithms verify in polynomial time whether  $x$  a valid solution is. The decision problem with this property forms the NP. In this work, we propose an NP-completeness result for the Steiner problem in coxeter graphs.

## 2. Preliminaries

### Definition 2.1

A **Steiner tree** is a tree in a distance graph which spans a given subset of vertices (Steiner point with the minimum total distance on its edges).

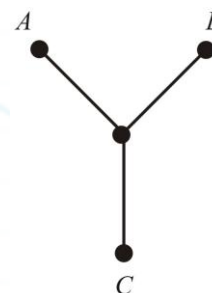


Figure 1: Steiner Tree

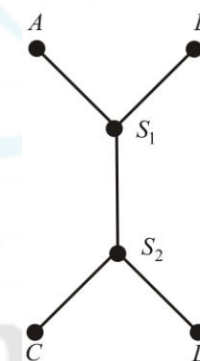


Figure 2: Steiner Tree

Two Steiner points  $S_1$  and  $S_2$ .

### The Steiner Tree problem

#### Definition

Let  $G = (V, E)$  be an evaluated graph. Let  $T$  be a set of terminal nodes that should be connected. The Steiner problem consists of finding a tree of  $G$  containing all terminal nodes  $T$  with a minimum weight. The optimal tree can contain other nodes called Steiner nodes in the set  $= T \setminus V$ . We note that two special cases of Steiner problem are solved polynomially.

If  $|T| = 2$ , then the Steiner problem is equivalent to shortest path.

If  $T = V$  then the Steiner problem is equivalent to the minimum spanning tree problem.

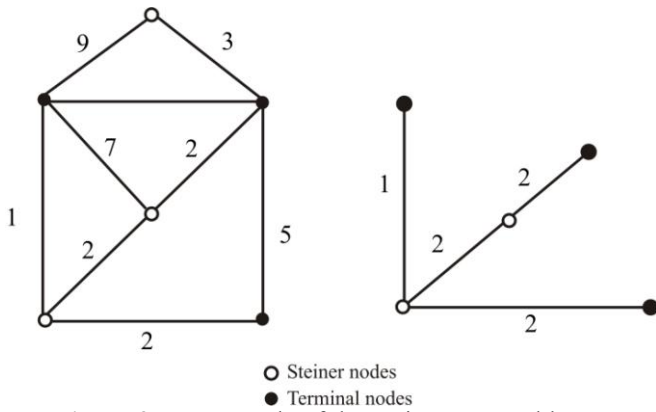


Figure 3: An example of the Steiner tree problem

**Definition 2.2**

Let a connected graph  $G = (V, E)$  and a set  $K \subseteq Y$  of terminals, then the **Steiner minimum tree** for  $K$  in  $G$  that is Steiner tree  $\tau$  for  $K$  such that

$$|E(T)| = \min \{E(T') / T' \text{ is a steinertree for } K \text{ in } G\}$$

In the Steiner minimum tree problem, the vertices are divided into two parts, terminals and non-terminal vertices. The terminals are the given vertices which must be included in the solution.

**Example 2.2**

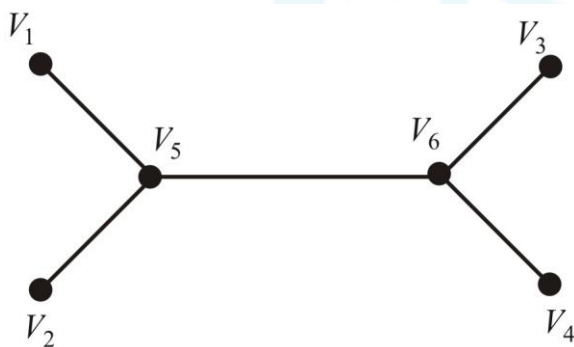


Figure 4: Steiner minimal tree

$V_1, V_2, V_3, V_4$  are terminals  $V_5$  and  $V_6$  is non-terminals.

**Definition 2.3**

A class of problems solvable by non deterministic polynomial time algorithm is called **NP**.

**Definition 2.4**

A problem is **NP-complete** if

1. It is an element of the class NP.
2. Another NP-complete problem is polynomial time reducible to it.

**Definition 2.5**

A Steiner minimum tree for  $K$  is given such that some of the terminals are interior points. Then we decompose this tree into components so that terminals only occur as leaves of these components. Such a component is called **full component**.

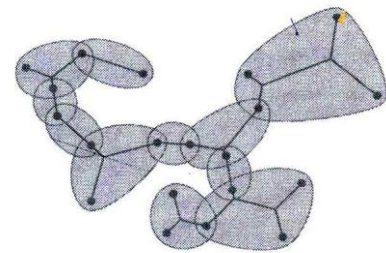


Figure 5: The full components of a Steiner Tree

**Definition 2.6**

The **Coxeter graph** is a non-hamiltonian cubic symmetric graph on 28 vertices and 42 edges.

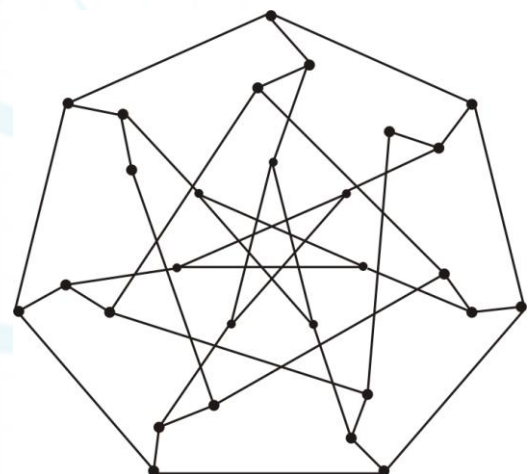


Figure 6: The Coxeter graph

**Properties of the Coxeter graph**

1. A spanning cycle in a graph is called a Hamiltonian cycle.
2. A graph having a Hamiltonian cycle is called a Hamiltonian graph.
3. The coxeter graph has no Hamiltonian cycle

This implies coxeter graph is not Hamiltonian.

4. A regular graph of degree 3 is called cubic graph.
5. Each vertex of a coxeter graph is of degree 3.

This implies coxeter graph is called cubic graph.

**Definition 2.7**

A **symmetric graph** is a graph in which both edge and vertex are transitive.

**Definition 2.8**

An **edge transitive graph** is a graph such that any two edges are equivalent under some element of its automorphism group.

**Example**

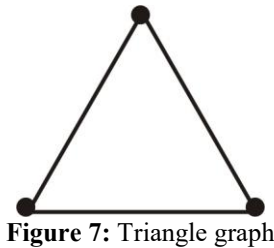


Figure 7: Triangle graph

More precisely, a graph is edge transitive if for all pairs of edges  $(e_1, e_2)$  there exists an element  $\gamma$  of the edge automorphism group  $Aut(G)$  such that  $\gamma(e_1) = e_2$ . An undirected graph is edge transitive iff its line graph is vertex transitive.

**Definition 2.9**

A **vertex – transitive graph** is a graph such that every pair of vertices is equivalent under some element of its automorphism group.

**Example**

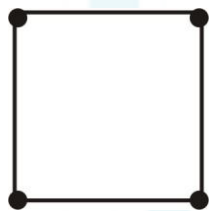


Figure 8: Square graph

**Steiner Problem in Coxeter Graph Which is NP-Complete**

**Result 1**

Steiner problem in Coxeter graph which is NP-complete.

**Proof**

Let the Steiner problem in graph is  $\in$  NP, it is sufficient to show that Steiner problem in Coxeter graph is NP-complete.

To construct a Coxeter graph on 28 vertices and 42 edges.

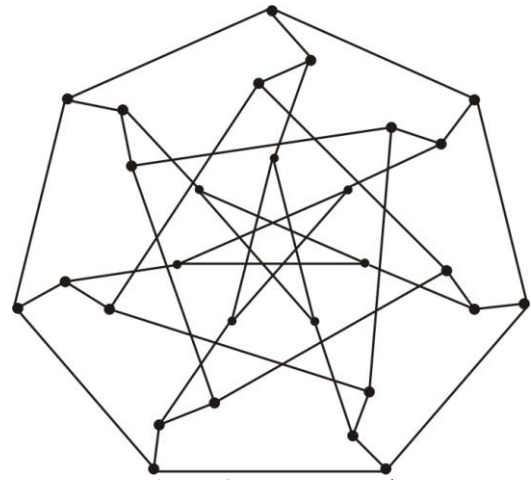


Figure 9: Coxeter graph

Now we take a Steiner tree from the Coxeter graph of maximum number of vertices in the following Figure 10.

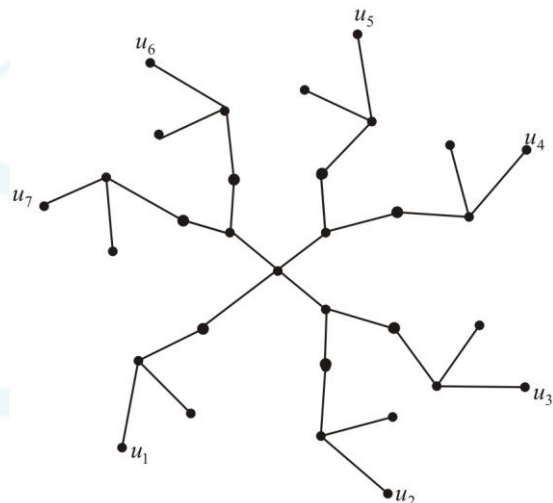
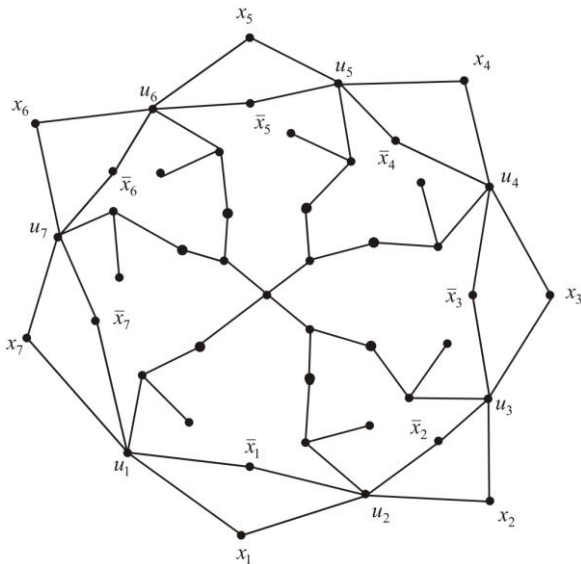


Figure 10: A Steiner tree in Coxeter graph

We reduce 3SAT to Steiner problem in Coxeter graphs. Let  $x_1, x_2, x_3, \dots, x_n$  be the variables  $c_1, c_2, \dots, c_n$  the clauses in an arbitrary instance of 3 SAT.

Our aim is to construct a Coxeter graph  $G = (V, E)$  and a terminal set  $K$ , and a bound  $B$  such that Coxeter contains Steiner tree  $T$  for  $K$  at size at most  $B$  if and only if the given 3 SAT instance is satisfiable.

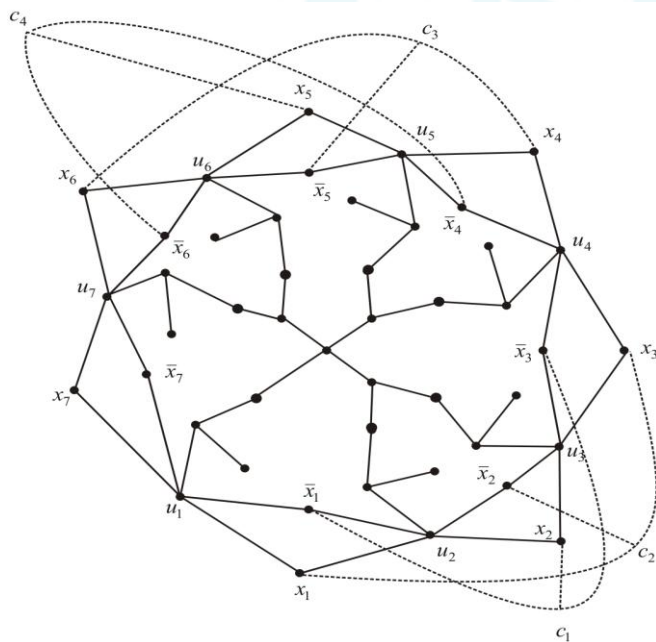
Transforming 3 SAT to Steiner problem in Coxeter graph is constructed as follows. We connect  $u_1$  and  $u_7$  by a variable path in Figure 11.



**Figure 11:** Transforming 3SAT to Steiner Problem in Coxeter graph

Then we create for every clause gadget consisting of a vertex  $C_i$  that is connected to the literals contained in the clause  $C_i$  by paths of Length  $t = 2n + 1$ .

As terminal set  $K = \{u_1, u_7\} \cup \{c_1, c_2, c_3, c_4\}$  and set  $B = 2n + t.m$



**Figure 12:** The clause gadget for the clauses

$$C_1 = \bar{x}_1 \vee x_2 \vee \bar{x}_3$$

$$C_2 = x_3 \vee \bar{x}_2 \vee x_1$$

$$C_3 = x_4 \vee \bar{x}_5 \vee x_6$$

$$C_4 = \bar{x}_4 \vee x_5 \vee \bar{x}_6$$

The dashed lines indicated paths of Length  $t = 2n + 1$  from  $C_i$  to the appropriate vertices on the variable path.

Let  $x_i \in P$  if  $x_i$  is said to true in this assignment and  $\bar{x}_i \in P$  otherwise. To construct a Steiner tree for  $K$  we start with  $u_1 - u_7$  path reflecting a satisfying assignment.

$x_i$  is true for variables. Hence we arrive from our SAT problem seven variable and 4 clauses for a 3SAT problem.

The number of variables  $n = 7$  to form the clauses  $\{c_1, c_2, c_3, c_4\}$  and the terminal set  $K = \{u, v\} \cup \{c_1, c_2, c_3, c_4\}$  and  $B = 2n + t.m$  Next observe that every clause the vertex  $C_i$  can be connected by path of Length.

$$t = 2n + 1$$

$$n = 7 \Rightarrow t = 2(7) + 1 = 15$$

In this way we obtain a Steiner tree for  $K$  of Length  $B = 2n + t.m$

$$\begin{aligned} m = 4 &\Rightarrow B = 2n + t.m \\ &= 2(7) + 15.4 \\ &= 14 + 60 \\ &= 74 \end{aligned}$$

A Steiner tree for this Coxeter graph we starting with a  $u_1 - u_7$  path  $P$  reflecting a satisfying assignment.

On the other hand, we assume now that  $T$  is a Steiner tree for  $K$  of Length at most  $B$ , Trivially for each clause to the vertex  $C_i$  has to be connected to the variable path.

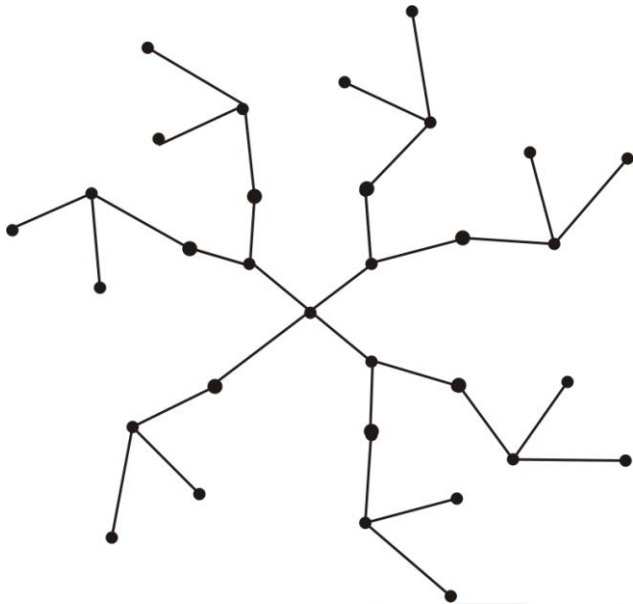
$$\begin{aligned} \text{Then } |E(T)| &\geq (m + 1).t > B \\ &\geq (4 + 1).15 > 74 \\ &\geq 5.15 > 74 \\ &\geq 75 > 74 \Rightarrow \text{Contradiction} \end{aligned}$$

In this graphs  $u_1 - u_7$  path contains 28 edges and that each clause gadget is connected to this path using exactly  $t$  edges.

This shows that  $u_1$  and  $u_7$  can only be connected along the variable path, which requires atleast  $2n$  edges.

In this Coxeter graph  $u_1 - u_7$  Path contains 28 edges. Thus  $u_1 - u_7$  path reflects a satisfying assignment. This implies that a Steiner problem in Coxeter graph is NP-complete.

In figure 10 the Steiner tree contains 14 terminals.



**Figure 13:** A Steiner tree in Coxeter graph

This implies that every full component of a Coxeter graph contains at most four terminals. This implies that R-Restricted Steiner problem in Coxeter graph is NP-complete.

#### Result 1

Steiner problem in Coxeter graph is NP-complete.

#### Result 2

Every  $u_1 - u_7$  path of Steiner tree in Coxeter graph is NP-complete and every  $u_1 - u_7$  path of Coxeter graph contains exactly  $2n$  edges.

#### Result 3

A Steiner tree of Coxeter graph, every full component contains at most 4 terminals.

#### Result 4

Transforming 3SAT to Steiner problem in Coxeter graph is NP-complete.

### 3. Conclusion

In this paper, we proved that the Steiner tree problem in Coxeter graph which is NP-complete. We have also shown that every full component of a Steiner tree contains almost 4 terminals and every  $u - v$  path of Steiner tree in Coxeter graph contains exactly  $2n$  edges.

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