Prime Difference Speed Sequence Graceful Graphs

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Abstract: The idea of prime difference speed sequence graphs are imposed on Fibonacci sequence. It is tested on various graphs like path, cycles, Kn, Knp, gear graph, D(Tn), fan graph, star graph, wheel graph etc. and the graceful labeling of those graphs are obtained. The strongly prime difference speed sequence graphs are also obtained.

Keywords: graceful labeling, difference speed sequence labeling, speed sequence graphs, prime difference speed sequence graphs, etc.

AMS Subject Classification: 05C78

1. Introduction

Throughout this paper, by a graph we mean a finite, undirected graph G(V, E) with 'p' vertices and 'q' edges. A detailed survey of graph labeling can be found in the dynamic survey of labeling by J.A. Gallian. In this paper we introduce a new labeling called prime difference speed sequence labeling.

Definition 1.1
Let G = (V(G), E(G)) be a graph with p vertices. A bijection f : V(G) → {1, 2, ..., p} is called a prime labeling if for each edge e = uv, gcd(f(u), f(v)) = 1. A graph which admits a prime labeling is called a Prime graph.

Definition 1.2
In a graph G with q edges, if f is an injection from the vertices of G to the set {0, 1, 2, ..., q}, then we call the graph as a Fibonacci graceful graph.

Definition 1.3
A (p, q) graph G(V, E) is said to be a difference speed sequence graceful graph if there exists a bijection f : V(G) → {0, 1, 2, ..., q} such that the induced mapping f : E(G) → |Δ(x)|/i = 1, 2, 3, ..., n defined by f(uv) = |f(u) − f(v)| is a bijection. Here Δ(x) = (Δn, x) = (x1, x3, x5, ..., x2n−1) and (x) is the Fibonacci sequence. The function f is called a difference speed sequence graceful graph.

Definition 1.4
A (p, q) graph G(V, E) is said to be a prime difference speed sequence graceful graph if there exists a bijection f : V(G) → {Δ(x)/i = 1, 2, 3, ..., n} and for each edge e = uv, gcd(f(u), f(v)) = 1. Here Δ(x) = (Δn, x) = (x1, x3, x5, ..., x2n−1) and (x) is any sequence. A graph which admits Prime difference speed sequence labeling is called a Prime difference speed sequence graph.

Definition 1.5
Let u and v be two different vertices of graph G. A new graph Guv is constructed by identifying (fusing) two vertices u and v by a single new vertex x such that every edge which was incident with either u or v in G is now incident with x in Guv.

Definition 1.6
A Gear graph is a graph obtained from Wheel graph, with a vertex added between each pair of adjacent vertices of an outer cycle.

Definition 1.7
A triangular snake Tn is obtained from a path u1, u2, ..., un by joining ui and ui+1 to a new vertex vi for 1 ≤ i ≤ n − 1.

Definition 1.8
A Double triangular snake D(Tn) consists of two triangular snakes that have a common path.

2. Prime Difference Sequence Labeling of Some Graphs by Fibonacci Numbers

Theorem 2.1
The Path Pn is a Prime difference speed sequence graph.

Proof:
Let V(Pn) = {u1, u2, ..., un} where ui = (Δixk) where (xk) is a Fibonacci sequence.

When we label the vertices for the path using difference speed sequence, we see that the consecutive adjacent vertices having labels in such a manner that gcd(f(u), f(v)) = 1.

Here f(u1) = (x1 − x3), f(u2) = (x2 − x4), f(u3) = (x3 − x5), ..., f(un−1) = (xn−1 − x1+n), such that it satisfies the condition gcd(f(u), f(v)) = 1.

Therefore Pn is a Fibonacci Prime difference speed sequence graph.

Theorem 2.2
The cycle Cn is a Prime difference speed sequence graph.
Proof:
Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$

The same labeling pattern is followed as in the path for both the cases $I$ when $n$ is odd and $II$ when $n$ is even.

Let $u_1, u_2, \ldots, u_n$ be the $n$ vertices.

The vertices are labeled such that $f(u_1) = (x_1 - x_2), f(u_2) = (x_2 - x_3), f(u_3) = (x_3 - x_4), \ldots, f(u_n) = (x_1 - x_{i+2})$, such that it satisfies $\gcd(f(u), f(v)) = 1$.

**Theorem 2.3**
The complete graph $K_n$ is not a Prime difference speed sequence graph for $n \geq 4$.

**Proof:**
Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$

$|E(G)| = \frac{n(n-1)}{2}$

When we assign labels the condition $\gcd(f(u), f(v)) = 1$ is satisfied for $n = 1$ to 4.

When $n \geq 4$, for at least any one of the edges does not satisfy the required result. Therefore $K_n$, $n \geq 4$ is not a prime difference speed sequence graph.

**Example 2.5**
The Prime difference speed sequence labeling of the graph obtained by identifying the apex vertex with a pendant vertex of $K_{1,9}$.

Since the self-loop is counted twice, the $\deg(u_0) = 8 + 2 = 10$.

**Example 2.6**
The Prime difference speed sequence graph labeling obtained by identifying the apex vertex with two pendant vertices with a pendant vertex of $K_{1,9}$.

Suppose we fix $f(v_i) = (x_{1 - x_{i+2}})$, for all $i = 0, 1, \ldots, n$.

Then the vertices $v_0$ and $v_{n-1}$ are identified.

In all the pendant vertices $\deg(v_i) = 1$ for all $i = 2$ to $n-2$, $\deg(u_0) = 2$, $\deg(v_0) = 9$. 

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**Case 1:** The apex vertex $v_0$ is identified with any of the pendant vertices (say $v_i$).

Let the new vertex be $u_0$ and the resultant graph be $G$. 

Then $\deg(v_i) = 1$, for $i = 2, 3, \ldots, n$ and $\deg(u_0) = n + 1$ as there is a loop incident at $u_0$.

Define $f: V(G) \rightarrow \{\Delta(x)/i = 1, 2, 3, \ldots n\}$ as $f(v_i) = i$ for $i = 2, 3, \ldots, n$ and $f(u_0) = 1$.

Clearly $f$ is an injection and $\gcd(f(u), f(v)) = 1$ for every pair of adjacent vertices $u$ and $v$ of $G$.

Hence $G$ is a Prime difference speed sequence graph.

**Case 2:** Any two of the pendant vertices (say $v_{n-1}$ and $v_n$) are identified.

Let the new vertex be $u_0$, and the resultant graph be $G$.

So, in $G$, $\deg(v_i) = 1$, for $i = 1, 2, \ldots, n-2$, $\deg(u_{n-1}) = 2$ and $\deg(v_0) = n$.

Define $f: V(G) \rightarrow \{\Delta(x)/i = 1, 2, 3, \ldots n\}$ as $f(v_i) = i + 1$ for $i = 0, 1, 2, \ldots, n-2$ and $f(u_{n-1}) = n$.

Obviously $f$ is an injection and $\gcd(f(u), f(v)) = 1$ for every pair of adjacent vertices $u$ and $v$ of $G$.

Hence $G$ is a Prime difference speed sequence graph.
Theorem 2.7
The graph obtained by identifying two vertices with label 1 and \( p \) of prime difference speed sequence graph is also a prime difference speed sequence graph if \( p \) is a prime and \( G \) is a prime labeling of \( G \).

Proof:
Assume that \( f \) is a prime labeling of \( G \).
Assign \( a \) as the label for the vertex \( v_a \) for \( a = 1 \) to \( p \).
Then the new vertex of the graph \( G' \) will be \( u_1 \) which is obtained by identifying \( v_1 \) and \( v_p \) of \( G \).
We define \( f_1: \{ u_1, v_2, v_3, \ldots, v_{p-1} \} \to \{ 1, 2, \ldots, p-1 \} \) as
\[
 f_1(x) = \begin{cases} 
 f(v_a) & \text{if } x = v_a, a = 2, \ldots, p-1 \\
 1 & \text{if } x = u_1 
\end{cases}
\]
Then \( f_1(x) = \begin{cases} 
 a & \text{if } x = v_a, a = 2, \ldots, p-1 \\
 1 & \text{if } x = u_1 
\end{cases}
\]
Clearly \( f \) is an injection.

We claim that \( \gcd(f(u), f(v)) = 1 \) for any arbitrary edge \( e = uv \) of \( G' \). Then the following cases arise. To prove our claim, the following cases are to be considered.

Case 1
If \( u = u_1 \), then 
\[
 \gcd(f(u), f_1(f(v))) = \gcd(f(u_1), f_1(f(v))) \]
\[
 = \gcd(1, f_1(f(v))) 
\]
\[
 = 1 
\]

Case 2
If \( u \neq u_1 \), and \( v = u_1 \), then 
\[
 \gcd(f(u), f_1(f(v))) = \gcd(f(u), f_1(u_1)) \]
\[
 = \gcd(f(u), 1) 
\]
\[
 = 1 
\]

Case 3
If \( u \neq u_1 \), and \( v \neq u_1 \), then \( u = v_a \), \( v = v_b \) for some \( a, b = 2, 3, \ldots, p - 1 \) with \( a \neq b \)
then \( \gcd(f_1(u), f_1(f(v))) = \gcd(f(v_a), f(v_b)) \)
\[
 = \gcd(f(v_a), f(v_b)) 
\]
\[
 = 1 
\]
as \( v_a \) and \( v_b \) are adjacent vertices in the prime graph \( G \).
Hence \( f_1 \) admits a prime difference speed sequence labeling for \( G' \).

Therefore \( G' \) is a prime difference speed sequence graph.

Example 2.8
In the following figures, the prime labeling of a graph \( G \) of order 6 and the prime labeling for the graph \( G' \) obtained by identifying the vertices of \( G \) with label 1 and 7 are shown.

Theorem 2.9
The graph obtained by identifying any two vertices of \( P_n \) is a Prime difference speed sequence graph.

Proof:
Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of \( P_n \).
Let \( u \) be the new vertex of a graph \( G \) obtained by identifying two distinct vertices \( v_a \) and \( v_b \) of \( P_n \).

Then \( G \) is nothing but a cycle or loop with at most two paths attached at \( u \). Such a graph is a Prime difference speed sequence graph.

Example 2.10
The various graphs obtained by identifying any two vertices of path \( P_n \) are as follows.

A prime difference sequence labeling of \( P_3 \)

A prime difference sequence labeling of the graph obtained by identifying \( v_t \) and \( v_s \) of \( P_3 \)

A prime difference sequence labeling of the graph obtained by identifying \( v_t \) and \( v_s \) of \( P_5 \)

A prime difference sequence labeling of the graph obtained by identifying \( v_t \) and \( v_s \) of \( P_5 \)
A prime difference sequence labeling of the graph obtained by identifying $v_1$ and $v_3$ of $P_3$

**Theorem 2.11**
A Gear graph $G_r$, $r \geq 3$ is a Prime difference speed sequence graph.

**Proof:**
A Gear graph $G_r$, $r \geq 3$ has $2r + 1$ vertices and $3r$ edges. 
Step 1: Assign the central vertex as label 1 
Step 2: Remaining vertices in the outer cycle can be labeled as $2, 3, 5, \ldots$ upto the $n^{th}$ vertex.
Let $v_0$ be the apex vertex $v_1, v_2, v_3, \ldots, v_{2r+1}$ be the rim vertices. 
Let $f(u_0) = (x_1 - x_3), f(v_2) = (x_2 - x_4), \ldots \ldots, f(v_i) = (x_{2r+1} - x_{2r+3})$
so that $gcd(f(u), f(v)) = 1$ satisfying the prime difference speed sequence graph.

**3. Strongly Prime Difference Speed Sequence Graphs**

**Definition 3.1**
A graph $G$ is said to be a strongly prime difference speed sequence graph if for any vertex $v$ of $G$ there exists a prime difference speed sequence labeling $f$ satisfying $f(v) = 1$.

**Theorem 3.2**
The complete graph $K_n$ is not Strongly Prime difference speed sequence graph for $n > 4$.

**Proof:**
It is obvious that $K_1$ and $K_2$ are Strongly Prime difference speed sequence graph. 
Any vertex of $K_3$ and $K_4$ can be assigned the labels easily so that it satisfies the strongly prime difference speed sequence graph. 
Let $v_1, v_2, \ldots \ldots, v_n$ be the $n$ vertices of a graph. 
Let $n > 4$. 
Fix any vertex as $v_1$ and labeling the vertices consecutively, we see that atleast any one of the edge does not satisfy $gcd(f(u), f(v)) = 1$. 
Therefore $n > 4$ is not a Strongly Prime difference speed sequence graph.

**Theorem 3.3**
Every path is a Strongly Prime difference speed sequence graph.

**Proof:**
Let $v_1, v_2, \ldots \ldots, v_n$ be the consecutive vertices of $P_n$. 
If $v_a$ is any arbitrary vertex of $P_n$, then the following two cases arise:

Case 1:
If $v_a$ is either of the pendant vertices (say $v_a = v_1$) then the function $f: V(P_n) \rightarrow \{\Delta_i(x)/ i = 1, 2, 3, \ldots n\}$ defined by $f(v_1) = i, for all i = 1, 2, 3, \ldots n$ is a prime difference speed sequence labeling with $f(v_0) = f(v_1) = 1$.

Case 2:
If $v_a$ is not a pendant vertex then $a = j$ for some $j \in \{2, 3, \ldots, n - 1\}$ then the function $f: V(P_n) \rightarrow \{\Delta_i(x)/ i = 1, 2, 3, \ldots n\}$ is a prime difference speed sequence labeling with $f(v_a) = f(v_j) = 1$.

Thus $P_n$ is a strongly Prime difference speed sequence graph.

**Example 3.4**
By assigning label 1 to any arbitrary vertex of path $P_a$, we get different strongly prime difference speed sequence graphs.

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**A strongly prime difference sequence labeling of $P_4$ having $v_1$ as label 1:**

**A strongly prime difference sequence labeling of $P_4$ having $v_2$ as label 1:**

**A strongly prime difference sequence labeling of $P_4$ having $v_3$ as label 1:**

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**International Journal of Scientific Engineering and Research (IJSER)**

**ISSN (Online): 2347-3878**


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**www.ijser.in**

**Volume 5 Issue 1, January 2017**

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**Paper ID: IJSER151192**

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