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# Application of Equimomental System of Point Masses for Dynamic Balancing of Mechanisms

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**Abstract:** ***This paper presents the concept of equimomental system of point-masses for rigid body to balance the mechanisms dynamically. The links of mechanism are modelled as rigid bodies for kinematic and dynamic analysis. The mass and moment of inertias of the links govern the shaking force and shaking moment transmitted to the frame on which they are mounted. Optimization of mass and moment of inertias methodology is used in minimizing the shaking force and shaking moment. The formulation of optimization problem is greatly simplified using the equimomental system of point-masses. The effectiveness of the method is illustrated with an example.***

**Keywords:** Equimomental system, Dynamic balancing, Shaking force, Shaking moment, Mechanisms

## 1. Introduction

A rigid body can be defined as a system of particles where the distances between particles remain essentially unchanged. However, this is an idealization as all solid bodies change shape to some extent when the forces are applied to them. Moreover, if the movements associated with the changes in shape are very small compared with the overall movements of the body as a whole, then the ideal concept of rigidity is quite acceptable. The machine mechanisms, land and air vehicles, rockets and spacecrafts, and many moving structures can be analysed using the concept of the rigid body [1-3].

To balance a mechanism, one has to eliminate the shaking force and shaking moment transmitted to the ground. The links of such mechanism can be modelled as rigid bodies for simplifying the kinematic and dynamic analysis [4]. The mechanisms are to be balanced either statically or dynamically. In some cases, static balancing can be acceptable substitute for dynamic balancing and is generally easier to do.

Like any system of forces acting on a rigid body can be replaced by an equivalent system of forces which produce identical motion, an equivalent mass distribution of a rigid body can be determined. For example, spatial mass distribution of a rigid body can be converted into a system of point-masses keeping the dynamic behavior identical. Such dynamically indistinguishable systems are called equimomental systems. The general requirements for the dynamical equivalence were laid down by Routh [5]. The set of point-masses and the rigid body are equimomental if they have the same total mass, the same center of mass, and the same inertia tensor with respect to the same coordinate frame [6]. However, there is no such limit on the maximum number of point-masses. The number of parameters related to the point-masses increase with increase of point- masses. It is shown that a set of seven point-masses is very effective in reducing shaking force and shaking moment in the mechanism [7]. This set of rigidly connected seven point-mass systems is explained in this paper to balance the mechanism dynamically.

This paper is organized as follows. Section 2 explains the equations of motion for rigid body. Equations of motion for equimomental point-masses are re-written in section 3. Problem of minimizing shaking force and shaking moment for a rotating link is then formulated in Section 4. A numerical example is solved using the proposed method in section 5. Finally, conclusions are given in Section 6.

## 2. Equations of Motion of Rigid Body

A link of a multibody system is modelled as the rigid body for the dynamic analysis. The Newton-Euler equations of motion for the $i$th rigid body of a multibody system shown in the Figure 1 are expressed as [8]:

$$\mathbf{I}_i^c\dot{\boldsymbol{\omega}}_i + \boldsymbol{\Omega}_i\mathbf{I}_i^c\boldsymbol{\omega}_i = \mathbf{n}_i^c;\; m_i\dot{\mathbf{v}}_i^c = \mathbf{f}_i^c \quad (1)\text{-}(2)$$

Where $\mathbf{n}_i^c$ is resultant of pure moment and moment of external forces about the mass center, $C_i$, and $\mathbf{f}_i^c$ is resultant force acting on the body at $C_i$. Moreover, $\mathbf{I}_i^c$ is the centroidal inertia tensor with respect to $C_i$. In Eqs. (1) and (2), $m_i$, $\boldsymbol{\omega}_i$ and $\dot{\boldsymbol{\omega}}_i$ are defined as the mass, angular velocity and angular acceleration of the body. The three-dimensional vector $\dot{\mathbf{v}}_i^c$ defines the linear acceleration of the mass center.

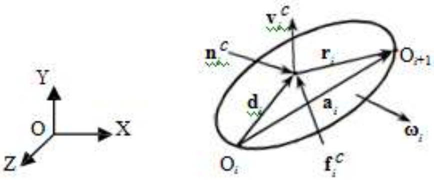


Figure 1 Free-body diagram of the $i$th body

Let the $i$th body is connected to the previous and next bodies at $O_i$ and $O_{i+1}$ through joints. The reference frame OXYZ is the fixed inertial frame. Then $\mathbf{a}_i$ can be defined as the link length. As in linkage balancing problem, we are more interested to know the forces at the joints, the equations with respect to mass center can be modified to find these forces directly at the joints. The NE equations, eqs. (1) and (2), are to be expressed with respect to $O_i$, in compact form as [7]:

$$\mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{E}_i \mathbf{t}_i = \mathbf{w}_i \quad (3)$$

Here, 6×6 matrices of extended mass, $\mathbf{M}_i$, extended angular velocity, $\mathbf{W}_i$, and $\mathbf{E}_i$ are defined as:

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{I}_i^o & m_i\mathbf{D}_i \\ -m_i\mathbf{D}_i & m_i\mathbf{1} \end{bmatrix};\ \mathbf{W}_i = \begin{bmatrix} \Omega_i & 0 \\ 0 & \Omega_i \end{bmatrix};\ \mathbf{E}_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};\ \text{where } \mathbf{E}_i = \begin{bmatrix} 0 & -\omega_{iz} & \omega_{iy} \\ \omega_{iz} & 0 & -\omega_{ix} \\ -\omega_{iy} & \omega_{ix} & 0 \end{bmatrix}; \quad (4)$$

Where $\mathbf{D}_i$ and $\mathbf{\Omega}_i$ are the 3x3 cross-product tensors associated with three-dimensional vectors $\mathbf{d}_i$ and $\boldsymbol{\omega}_i$, respectively. Note that **1** and **O** are 3×3 unity matrix and 3×3 zero matrix, respectively. Furthermore, the six-dimensional vectors of twist, $\mathbf{t}_i$, twist rate, $\dot{\mathbf{t}}_i$, and wrench $\mathbf{w}_i$, are defined as:

$$\mathbf{t}_i = \begin{bmatrix} \omega_i \\ \mathbf{v}_i \end{bmatrix};\quad \dot{\mathbf{t}}_i = \begin{bmatrix} \dot{\omega}_i \\ \dot{\mathbf{v}}_i \end{bmatrix};\quad \mathbf{w}_i = \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix} \quad (5)$$

where $\omega_i$, $\mathbf{v}_i$, $\mathbf{n}_i$ and $\mathbf{f}_i$, are angular velocity, linear velocity, resultant moment, and resultant force acting on the $i$th body, respectively, at $O_i$ of the body. Equation (3) is the desired NE equations of motion of the $i$th body written with respect to the point, $O_i$. It is suitable for finding the reaction forces at the joints. This formulation avoids the post processing for computing the reactions.

## 3. Equations of Motion for Equimomental System of Point-Masses

The Newton-Euler (NE) equations of motion for the rigid body presented in section 2 are, now, modified for the equimomental system of point-masses. A set of $n$ point-masses will be dynamically equivalent to the rigid body if [4]

$$\sum_{j=1}^{n} m_{ij} = m_i;\quad \sum_{j=1}^{n} m_{ij}x_{ij} = m_i\bar{x}_i;\quad \sum_{j=1}^{n} m_{ij}y_{ij} = m_i\bar{y}_i;\quad \sum_{j=1}^{n} m_{ij}z_{ij} = m_i\bar{z}_i \quad (6)\text{-}(9)$$

$$\sum_{j=1}^{n} m_{ij}x_{ij}y_{ij} = I_{ixy};\quad \sum_{j=1}^{n} m_{ij}y_{ij}z_{ij} = I_{iyz};\quad \sum_{j=1}^{n} m_{ij}z_{ij}x_{ij} = I_{izx} \quad (10)\text{-}(12)$$

$$\sum_{j=1}^{n} m_{ij}(y_{ij}^2+z_{ij}^2) = I_{ixx};\quad \sum_{j=1}^{n} m_{ij}(z_{ij}^2+x_{ij}^2) = I_{iyy};\quad \sum_{j=1}^{n} m_{ij}(x_{ij}^2+y_{ij}^2) = I_{izz} \quad (13)\text{-}(15)$$

here $m_{ij}$ is $j$th point-mass and $x_{ij}$, $y_{ij}$, $z_{ij}$ are its coordinates where as the rigid body has mass $m_i$, the mass center $(\bar{x},\bar{y},\bar{z})$ and the moment of inertias $I_{ixx}$, $I_{iyy}$, $I_{izz}$ and product of inertias $I_{ixy}$, $I_{iyz}$, $I_{izx}$. An equimomental system of seven point-masses as shown in Figure 2 is used here for the rigid body. Referring to Figure 2, the three-dimensional vectors, $\mathbf{d}_{ij}$ and $\mathbf{r}_{ij}$, are the positions the point-mass, $m_{ij}$, from the origins $O_i$ and $O_{i+1}$, respectively. Subscripts $i$ and $j$ denote the $i$th link and its $j$th point-mass, respectively.

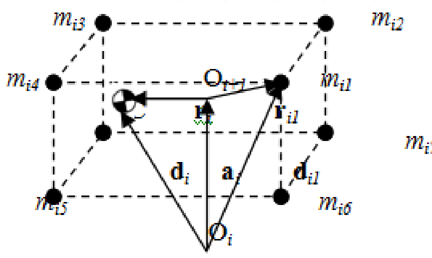


Figure 2 Equimomental system of seven point-masses

Vector $\mathbf{d}_i$ locating the total mass center in terms of $\mathbf{d}_{ij}$'s is obtained as:

$$\mathbf{d}_i = \frac{1}{m_i}\sum_{j=1}^{7} m_{ij}\mathbf{d}_{ij} \quad (16)$$

Denoting $\mathbf{d}_{ij}=[d_{ijx},\ d_{ijy},\ d_{ijz}]^T$, the 3x3 skew-symmetric matrix, $\mathbf{D}_i$, associated with the vector, $\mathbf{d}_i$, and inertia tensor, $\mathbf{I}_i^o$, about $O_i$, in terms of the point-mass parameters are represented as:

$$\mathbf{D}_i = \frac{1}{m_i}\begin{bmatrix} 0 & -\sum_{j=1}^{7} m_{ij}d_{ijz} & \sum_{j=1}^{7} m_{ij}d_{ijy} \\ \sum_{j=1}^{7} m_{ij}d_{ijz} & 0 & -\sum_{j=1}^{7} m_{ij}d_{ijx} \\ -\sum_{j=1}^{7} m_{ij}d_{ijy} & \sum_{j=1}^{7} m_{ij}d_{ijx} & 0 \end{bmatrix};$$

$$\mathbf{I}_i^o = \begin{bmatrix} \sum_{j=1}^{7} m_{ij}(d_{ijy}^2+d_{ijz}^2) & -\sum_{j=1}^{7} m_{ij}d_{ijx}d_{ijy} & -\sum_{j=1}^{7} m_{ij}d_{ijx}d_{ijz} \\ & \sum_{j=1}^{7} m_{ij}(d_{ijz}^2+d_{ijx}^2) & -\sum_{j=1}^{7} m_{ij}d_{ijy}d_{ijz} \\ sym & & \sum_{j=1}^{7} m_{ij}(d_{ijx}^2+d_{ijy}^2) \end{bmatrix} \quad (17)$$

Equations (16) and (17) define the mass matrix, $\mathbf{M}_i$, of the $i$th body in terms of the parameters of the equimomental seven point-masses. Putting this mass matrix $\mathbf{M}_i$ in the Eq. (3) will give the modified NE equations of motion for equimomental point- mass system.

## 4. Problem Formulation for Minimizing Shaking Force and Shaking Moment for a Rotating Link

Based on the analysis presented in sections 2 and 3, the problem for minimizing the shaking force and shaking moment for a rotating link is formulated in this section. The single link mechanism under consideration is shown in Figure 3.

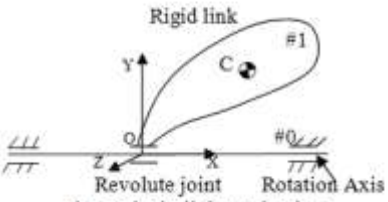


Fig. 3 Single link mechanism

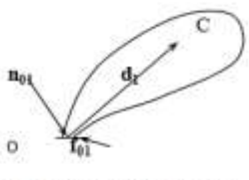


Fig. 4 Free-body diagram

The rigid link is connected to frame through a revolute joint and rotating about X axis. The fixed inertial frame, OXYZ, is located at the joint between link and frame. The path of center of mass of the link lies in YZ plane for complete cycle. The resultant moment about the origin, O, and the resultant force acting at O on the link are shown in Figure 4. Here the frame and the link are numbered as ≠0 and ≠1, respectively, for the analysis purpose. $\mathbf{f}_{01}$ and $\mathbf{n}_{01}$ are moments and forces applied by the frame on the link at the joint. Without losing generality, the external forces like gravity and dissipative forces are not considered here. As discussed in previous sections, this link can be treated as dynamically equivalent system of seven point-masses. A body fixed frame, $o_1x_1y_1z_1$ is suitably chosen for finding mass center and inertias. The point-masses and their locations are shown in Figure 5.

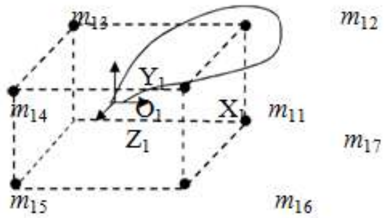


Figure 5 Equimomental seven point-masses

The positions of point-masses in body fixed frame, $o_1x_1y_1z_1$, are given by three- dimensional position vectors, $\mathbf{d}_{ij}$ $[d_{ijx}, d_{ijy}, d_{ijz}]^T$, for j=1,…,7. These position vectors are transferred into fixed inertial frame, OXYZ using two rotation matrices, $Q_\theta$ and $Q_\alpha$ about axis Z and X respectively, defined as:

$$\mathbf{Q}_\theta=\begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix};\quad \mathbf{Q}_\alpha=\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix};\quad \mathbf{Q}_i=\mathbf{Q}_\theta\mathbf{Q}_\alpha=\begin{bmatrix} C\theta & -S\theta C\alpha & S\theta S\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

where $C\theta$=cosθ and $S\theta$=sinθ, while θ and α are the angles of rotation about Z and X axes, respectively. Knowing the rotation matrix, $\mathbf{Q}_i$, the three-dimensional position vector $\mathbf{d}_{1j}$ and 3x3 inertia tensor, $\mathbf{I}_i^o$ for point-masses in frame $o_1x_1y_1z_1$ can be transformed into that of OXYZ as:

$$[\mathbf{d}_{1j}]_{OXYZ}=\mathbf{Q}_i[\mathbf{d}_{1j}]_{o_1x_1y_1z_1};\quad [\mathbf{I}_i^o]_{OXYZ}=\mathbf{Q}_i[\mathbf{I}_i^o]_{o_1x_1y_1z_1}\mathbf{Q}_i^T$$

The NE equations of motion of the link in the given mechanism as explained in Eq. (3) will be written as: $\mathbf{M}_1\dot{\mathbf{t}}_1+\mathbf{W}_1\mathbf{M}_1\mathbf{E}_1\mathbf{t}_1=\mathbf{w}_1$ (18)

The terms associated with Eq. (18) for single link mechanism are determined from Eq. (4) and Eq. (5) for $i$=1. For the problem under consideration:

$$\omega_1=[\omega_{1x}\ \ \omega_{1y}\ \ \omega_{1z}]^T=[\omega_{1x}\ \ 0\ \ 0]^T;\ \mathbf{v}_1=[v_{1x}\ \ v_{1y}\ \ v_{1z}]^T=[0\ \ 0\ \ 0]^T;\ \mathbf{n}_1=\mathbf{n}_{01}\ \text{and}\ \mathbf{f}_1=\mathbf{f}_{01} \tag{19}$$

Shaking force is defined as the reaction of the resultant inertia forces, whereas shaking moment about any particular point is the reactions of the resultant inertia couples and the moment of the inertia forces about that point [9]. For the mechanism under consideration, the shaking force and shaking moment with respect to the joint are obtained as:

$$\mathbf{f}_{sh}=-\mathbf{f}_{01};\ \mathbf{n}_{sh}=-\mathbf{n}_{01} \tag{20}$$

where three-dimensional vectors, $\mathbf{f}_{sh}$ and $\mathbf{n}_{sh}$ , represent shaking force and shaking moment while three-dimensional vectors, $\mathbf{f}_{01}$ and $\mathbf{n}_{01}$ are the vector components of $\mathbf{w}_1$. In this case, both shaking force and shaking moment transmitted to the frame are to be minimized, so the combined objective function is used for the optimization problem. For a combined objective function, its quantities are normalised with respect to the reference link parameters, which are defined as [9]:

$$\bar{\mathbf{f}}_{sh}=\frac{|\mathbf{f}_{sh}|}{ma\omega^2};\quad \bar{\mathbf{n}}_{sh}=\frac{|\mathbf{n}_{sh}|}{ma^2\omega^2} \tag{21}$$

The parameters $m$, $a$ and $\omega$ represent mass, length and angular velocity of the link respectively. The normalization of dynamic quantities allows one to consider the quantities of different dimensions during optimization [10]. The root mean square (RMS) values of the normalized shaking force and shaking moment are used for the minimization purpose. Considering the RMS values of the normalized shaking force and shaking moment, an optimality criterion is proposed as:

$$z=w_1\mathbf{f}_{sh,rms}+w_2\,\mathbf{n}_{sh,rms} \tag{22}$$

where $w_1$ and $w_2$ are the weighting factors. The seven point-masses parameters for the link are design variables for the optimization. The task here is to find the values of these variables to minimize the function, $z$.

## 5. Numerical Example

The effectiveness of the proposed method is shown using a cylindrical shaped link made of steel, whose parameters are given in table 1. The body fixed frame

$o_1x_1y_1z_1$ is chosen such that $x_1$ coincides with the axis of the cylinder. Mass and inertias of the normalized link are given in Table 2. MATLAB programs were developed for finding seven point-masses dynamically equivalent to the link. The resulting point- masses and their locations are shown in the Table 3.

**Table 1:** Parameters for rigid link in spatial motion

| Parameter | Description | Value |
|---|---|---|
| *m, a, R* | mass, length and radius of link | 61.23 kg, 1 m and |
| $\omega_x$, $\alpha$ | angular velocity and angle of rotation of link | 100 radians/sec, variable (0 to |
| $\theta$ | angle between x1 and X | 45° |
| $I_{xx}$ | moment of inertia of the link about the x-axis | 0.0765 kg-m$^2$ |
| $I_{yy}$= $I_{zz}$ | moment of inertia of the link about the y-axis | 20.4483 kg-m$^2$ |
| $I_{xy}$=$I_{yz}$=$I_{zx}$ | polar moment of inertias of the | 0 |
| *D* | distance of centre of mass from the origin of | $x_c$=0.5 m,$y_c$= $z_c$=0 |

**Table 2:** Mass and inertias of the normalized link

| *m* | $d_x$ | $d_y$ | $d_z$ | $I_{xx}$ | $I_{yy}$ | $I_{zz}$ | $I_{xy}$ | $I_{yz}$ | $I_{zx}$ |
|---|---|---|---|---|---|---|---|---|---|
| (kg) | (meter) | | | (kg-meter$^2$) | | | | | |
| 1 | 0.5 | 0 | 0 | 0.0765 | 20.4483 | 20.4483 | 0 | 0 | 0 |

**Table 3:** Equimomental point-masses of the normalized link

| $m_1$=$m_5$ | $m_2$=$m_6$ | $m_3$=$m_7$ | $m_4$ | $D_x$ | $d_y$=$d_z$ |
|---|---|---|---|---|---|
| (kg) | | | | (meter) | |
| 0.25 | 0.2165 | 0.03349 | 0 | 0.5774 | 0.0250 |

The point-masses are the design variables in this case and the optimization problem is solved using ''fmincon'' function of the optimization toolbox of MATLAB (Figure 6). The comparison of original and optimum values of shaking force and shaking moment components and resultant are shown in Figure 7 and 8, respectively. (discontinuous curves represent optimum values while continuous curves represent original values)

Design Variables: Point-masses (DV) ($m_1$,...,$m_7$)

Minimize $z = w_1 \mathbf{f}_{sh,rms} + w_2 \mathbf{n}_{sh,rms}$
Subject to $m^o \le m \le 5m^o$

Solution for DVs using fmincon Function of MATLAB Optimization Toolbox

**Figure 6:** Flowchart of MATLAB Algorithm

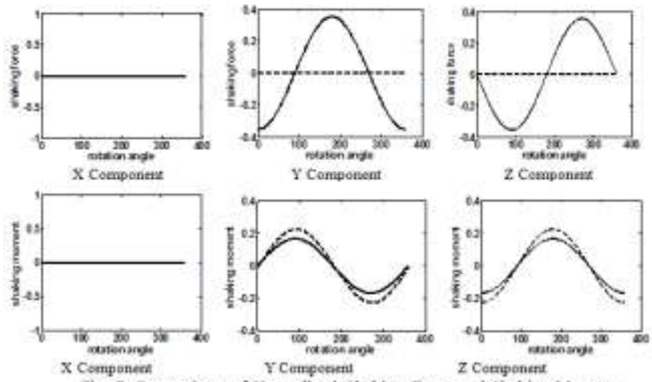


Fig. 7. Comparison of Normalized Shaking Force and Shaking Moment

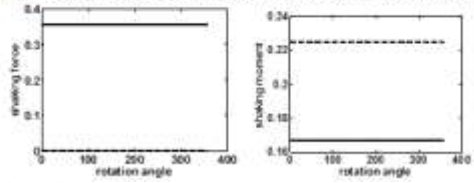


**Figure 8:** Comparison of Normalized Resultant Shaking Force and Shaking Moment

The resultant of the original as well as optimum shaking force and shaking moment values are shown in Table 4. Similarly, optimized balance mass, its location, and moment of inertia are given in Table 5.

**Table 4:** Comparison of the RMS values of the normalized shaking force and the shaking moment

| | Original Value | Optimized |
|---|---|---|
| Shaki | 0.3 | 1.4895 x 10$^{-6}$ |
| Shakin | 0.1 | 0.24 |

**Table 5:** Mass and inertia properties of the counterweight

| *M* | $d_x$ | $d_y$=$d_z$ | $I_{xx}$ | $I_{yy}$= | $I_{xy}$ | $I_{yz}$ | $I_{zx}$ |
|---|---|---|---|---|---|---|---|
| (kg) | (meter) | | (kg-meter$^2$) | | | | |
| 0.50 | -1.00 | 0 | -0.000625 | 0.1670 | -0.0046 | -0.000206 | -0.0046 |

From results, it is clear that x components of shaking force and shaking moment are zero for complete cycle as link is rotating about x axis. The optimum shaking force values for y and z components as well as resultant is reduced and reaches to zero. However, optimum shaking moment values are increased slightly as single link is considered in this problem. The reduction in shaking moment can be achieved for the mechanisms using this method. This methodology will be extended for the slider crank mechanism.

## 6. Conclusions

This paper presents a method for balancing the mechanism using the concept of the equimomental system for rigid body. The dynamic equations of motion are formulated systematically in the parameters related to the equimomental point-masses. Using these equations, the optimization problem is formulated for the balancing of shaking force and shaking moment of a rigid link in spatial motion. The proposed method is illustrated using a single link mechanism. The method completely balanced the shaking force transmitting to the frame. Further, the shape optimization may be used to find the shape and size of the counterweight.

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