International Journal of Scientific Engineering and Research (IJSER)
ISSN (Online): 2347-3878
Index Copernicus Value (2015): 62.86 | Impact Factor (2015): 3.791

$\tau_1\tau_2$-$\ddot{g}$-Closed sets in Bitopological Spaces

S. Vijayalakshmi[1], T. Indira[2]

[1]Research Scholar, PG & Research Department of Mathematics, Seethalakshmi Ramasami College, Trichy-2, India
[2]Associate Professor, PG & Research Department of Mathematics, Seethalakshmi Ramasami College, Trichy-2, India

Abstract: ***The aim of this paper is to introduce $\tau_1\tau_2$-$\ddot{g}$-Closed sets, $\tau_1\tau_2$-$\ddot{g}$-Open sets in Bitopological Spaces and to study about their properties.***

Keywords: $\tau_1\tau_2$-$\ddot{g}$-Closed sets,$\tau_1\tau_2$-$\ddot{g}$-Open sets

AMS Mathematics Subject Classification: 54E55

1. Introduction

Levine [8] introduced semi open sets in 1963 and also Levine [7] defined generalized closed sets in 1970. AblEl-Monsef et al.[2] introduced β-open sets. Veerakumar [11] introduced #g-closed sets in topological spaces. O.Ravi [10] and S.Ganesan [10] declared$\ddot{g}$-closed sets in topological spaces. Kelley [6] initiated the study of bitopological spaces in 1963. A nonempty set X equipped with two topologiesτ_1 and τ_2 is called a bitopological space and is denoted by (X, τ_1, τ_2). Since then several topologists generalized many of the results in topological spaces to bitopological spaces. Fukutake [3] introduced generalized closed sets in bitopological spaces. Fukutake [4] defined semi open sets in bitopological spaces.

2. Preliminaries

Definition 2.1: A subset A of a bitopological space (X, τ_1, τ_2) is called a

1. $\tau_1\tau_2$-semi open [2] if $A \subset \tau_2 cl(\tau_1 int(A))$ and it is called $\tau_1\tau_2$-semi closed [2] if $\tau_2 int(\tau_1 cl(A)) \subset A$
2. $\tau_1\tau_2$-pre open[5] if $A \subset \tau_2 int(\tau_1 cl(A))$ and $\tau_1\tau_2$-preclosed [5] if $\tau_2 cl(\tau_1 int(A)) \subset A$
3. $\tau_1\tau_2$-α-open [9] if $A \subset \tau_1 int\left(\tau_2 cl(\tau_1 int(A))\right)$.

Definition 2.2: A subset A of a bitopological space (X, τ_1, τ_2) is called a

1. $\tau_1\tau_2$-g-closed [3]($\tau_1\tau_2$-generalized closed) if τ_2-$cl(A) \subset U$, whenever $A \subset U$, U is τ_1-open.
2. $\tau_1\tau_2$-sg-closed [4]($\tau_1\tau_2$-semi generalized closed) if τ_2-$scl(A) \subset U$, whenever $A \subset U$, U is τ_1-semi open.
3. $\tau_1\tau_2$-gs-closed [5] ($\tau_1\tau_2$-generalized semi closed) if τ_2-$scl(A) \subset U$, whenever $A \subset U$, U is τ_1-open.
4. $\tau_1\tau_2$-αg-closed [9] ($\tau_1\tau_2$-α-generalized closed) if τ_2-$\alpha cl(A) \subset U$, whenever $A \subset U$, U is τ_1-open.
5. $\tau_1\tau_2$-gα-closed [9] ($\tau_1\tau_2$-generalized α-closed) if τ_2-$\alpha cl(A) \subset U$, whenever $A \subset U$, U is τ_1-α-open.
6. $\tau_1\tau_2$-$\hat{g}$-closed[5] if τ_2-$cl(A) \subset U$, whenever $A \subset U$, U is τ_1- semi open.

3. $\tau_1\tau_2$-$\ddot{g}$-Closed sets

Definition 3.1
A subset A of (X, τ_1, τ_2) is called a $\tau_1\tau_2$-$\ddot{g}$-closed if τ_2-$cl(A) \subset U$, whenever $A \subset U$ and$U$ is τ_1-sg-open in (X, τ_1).

Example 3.2
Let
$X = \{a, b, c\}$; $\tau_1 = \{X, \phi, \{a\}\}$; $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$.
Then $X, \phi, \{c\}, \{b, c\}$ are $\tau_1\tau_2$-$\ddot{g}$-Closed sets.

Theorem 3.3
Every τ_2-closed set is $\tau_1\tau_2$-$\ddot{g}$-closed sets.

Proof:
Let A be τ_2-closed. Then $\tau_2 - cl(A) = A. \Rightarrow \tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-sg-open.⇒A is $\tau_1\tau_2$-$\ddot{g}$-closed sets.

The converse of the above theorem need not be true as seen from the following example.

Example 3.4
Let $X = \{a, b, c, d\}$; $\tau_1 = \{X, \phi, \{a, b\}\}$; $\tau_2 = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$.
$\tau_1\tau_2$-$\ddot{g}$-closed sets are
$X, \phi, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}$.
Therefore $\{d\}, \{b, d\}$ are $\tau_1\tau_2$-$\ddot{g}$-closed set but they are notτ_2-closed.

Theorem 3.5
Every $\tau_1\tau_2$-$\ddot{g}$-closed set is $\tau_1\tau_2$-g-closed.

Proof:
Let $A \subseteq U$, U is τ_1-open. Then U is τ_1-sg-open.$\Rightarrow \tau_2 - cl(A) \subseteq U$.[∵ A is $\tau_1\tau_2$-$\ddot{g}$-closed]$\Rightarrow \tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-open.⇒A is $\tau_1\tau_2$-g-closed sets.

The converse of the above theorem need not be true as seen from the following example.

Example 3.6
Let $X = \{a, b, c, d\}$; $\tau_1 = \{X, \phi, \{a, b\}\}$; $\tau_2 = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$

$\tau_1\tau_2$-$\ddot{g}$-closed sets are
$X, \phi, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}$
$\tau_1\tau_2$-g-closed sets are
$X, \phi, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
Here $\{c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}$ is $\tau_1\tau_2$-g-closed set but they are not$\tau_1\tau_2$-$\ddot{g}$-closed set.

Theorem 3.7
Every $\tau_1\tau_2$-$\ddot{g}$-closed set is $\tau_1\tau_2$-$\hat{g}$- closed.

Proof:
Let $A \subseteq U$, U is τ_1-semi open. Then U is τ_1-sg-open. Since A is $\tau_1\tau_2$-$\ddot{g}$-closed,$\tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-semi open.$\Rightarrow \tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1- semi open.⇒A is $\tau_1\tau_2$-$\hat{g}$-closed sets.

Theorem 3.8
Every $\tau_1\tau_2$-$\ddot{g}$-closed set is $\tau_1\tau_2$-gs-closed.

Proof:
Let $A \subseteq U$, U is τ_1-open. U is τ_1-open$\Rightarrow U$is τ_1-sg-open.$\Rightarrow \tau_2 - cl(A) \subseteq U$ (By assumption)But $\tau_2 - scl(A) \subseteq \tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-open.$\Rightarrow \tau_2 - scl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-open.⇒ A is $\tau_1\tau_2$-gs-closed.

Theorem 3.9
Every $\tau_1\tau_2$-$\ddot{g}$-closed set is $\tau_1\tau_2$-αg-closed.

Proof:
Let $A \subseteq U$, U is τ_1-open. U is τ_1-open$\Rightarrow U$ is τ_1-sg-open.$\Rightarrow \tau_2 - cl(A) \subseteq U$ (By assumption) But $\tau_2 - \alpha cl(A) \subseteq \tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-open.$\Rightarrow \tau_2 - \alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-open.
⇒ A is $\tau_1\tau_2$-αg-closed.

Theorem 3.10
Every $\tau_1\tau_2$-$\ddot{g}$-closed set is $\tau_1\tau_2$-gα-closed.

Proof:
Let $A \subseteq U$, U is τ_1-α-open. U is τ_1-α-open$\Rightarrow U$ is τ_1-sg-open.
$\Rightarrow \tau_2 - cl(A) \subseteq U$ (By assumption)But $\tau_2 - \alpha cl(A) \subseteq \tau_2 - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-α-open.$\Rightarrow \tau_2 - \alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is τ_1-α-open.⇒ A is $\tau_1\tau_2$-gα-closed.

Theorem 3.11
Union of two $\tau_1\tau_2$-$\ddot{g}$-closed sets is $\tau_1\tau_2$-$\ddot{g}$-closed.

Proof:
Assume that A and B are $\tau_1\tau_2$-$\ddot{g}$-closed sets.Let $A \cup B \subseteq U$ where U is τ_1-sg-open. Then $A \subseteq U$ and $B \subseteq U$.$\Rightarrow \tau_2 - cl(A) \subseteq U$ and $\tau_2 - cl(B) \subseteq U \Rightarrow \tau_2 - cl(A) \cup \tau_2 - cl(B) \subseteq U$But $\tau_2 - cl(A \cup B) = \tau_2 - cl(A) \cup \tau_2 - cl(B) \subseteq U$. Therefore $\tau_2 - cl(A \cup B) \subseteq U$ whenever $A \cup B \subseteq U$&U is τ_1-sg-open. Therefore $A \cup B$ is $\tau_1\tau_2$-$\ddot{g}$-closed.

Remark 3.12
The intersection of two $\tau_1\tau_2$-$\ddot{g}$-closed sets need not be $\tau_1\tau_2$-$\ddot{g}$-closed.
This can be seen from the following example.

Example 3.13
Let $X = \{a, b, c\}$; $\tau_1 = \{X, \phi, \{b, c\}\}$; $\tau_2 = \{X, \phi, \{a\}\}$
$\tau_1\tau_2$-$\ddot{g}$-closed sets $X, \phi, \{a\}, \{a, b\}, \{b, c\}, \{a, c\}$.
$A = \{a, b\}$ and $B = \{b, c\}$
$A \cap B = \{b\}$ is not $\tau_1\tau_2$-$\ddot{g}$-closed set.

Theorem 3.14
Let A be $\tau_1\tau_2$-$\ddot{g}$-closed and $A \subset B \subset \tau_2 - cl(A)$ then B is $\tau_1\tau_2$-$\ddot{g}$-closed.

Proof:
Let $B \subseteq U$ where U is τ_1-sg-open.Then $A \subset B \subset U \Rightarrow \tau_2 - cl(A) \subseteq U$.
Given $B \subset \tau_2 - cl(A)$ and $\tau_2 - cl(B)$ is the smallest closed set containing B.
$\therefore B \subset \tau_2 - cl(B) \subset \tau_2 - cl(A) \subset U \Rightarrow \tau_2 - cl(B) \subset U \Rightarrow$ B is $\tau_1\tau_2$-$\ddot{g}$-closed.

Theorem 3.15
If A is $\tau_1\tau_2$-$\ddot{g}$-closed then $\tau_2 - cl(A) - A$ does not contains any non-empty τ_1-sg-closed set.

Proof:
Suppose $\tau_2 - cl(A) - A$ contains a not empty τ_1-sg-closed set F.
Then $F \subset \tau_2 - cl(A) - A.\Rightarrow F \subset \tau_2 - cl(A)$ but $F \not\subset A \Rightarrow F \subset A^c \Rightarrow A \subset F^c$ where F^c is τ_1-sg-open.$\Rightarrow \tau_2 - cl(A) \subset F^c \Rightarrow F \subset \left(\tau_2 - cl(A)\right)^c$.
We have $F \subset \tau_2 - cl(A) \cap \left(\tau_2 - cl(A)\right)^c = \phi \therefore \tau_2 - cl(A) - A$ does not contains any non-empty τ_1-sg-closed set.

Theorem 3.16
Let A be $\tau_1\tau_2$-$\ddot{g}$-closed. Then A is τ_2-closed iff $\tau_2 - cl(A) - A$ is τ_1-sg-closed set.

Proof:
Suppose that A is $\tau_1\tau_2$-$\ddot{g}$-closed and τ_2-closed. Then $\tau_2 - cl(A) = A$.
$\Rightarrow \tau_2 - cl(A) - A = \phi$. which is τ_1-sg-closed set.
Conversely, assume that A is A be $\tau_1\tau_2$-$\ddot{g}$-closed and $\tau_2 - cl(A) - A$ is τ_1-sg-closed set.Since A is is $\tau_1\tau_2$-$\ddot{g}$-closed, $\tau_2 - cl(A) - A$ does not contains any non-empty τ_1-sg-closed set.$\Rightarrow \tau_2 - cl(A) - A = \phi \Rightarrow \tau_2 - cl(A) = A$.⇒A is τ_2-closed.

Theorem 3.17
If A is $\tau_1\tau_2$-$\ddot{g}$-closed and $A \subset B \subset \tau_2 - cl(A)$ then $\tau_2 - cl(B) - B$ contains no non-empty τ_1-sg-closed set.

Proof:
Let A be $\tau_1\tau_2$-$\ddot{g}$-closed and $A \subset B \subset \tau_2 - cl(A)$. By theorem 3.14, B is $\tau_1\tau_2$-$\ddot{g}$-closed. Since B is $\tau_1\tau_2$-$\ddot{g}$-closed, then by theorem 2.15 $\tau_2 - cl(B) - B$ contains no non-empty τ_1-sg-closed set.Hence $\tau_2 - cl(B) - B$ contains no non-empty τ_1-sg-closed set.

Theorem 3.18
For each $x \in X$, the singleton $\{x\}$ is either τ_1-sg-closed set or its complement $\{x\}^c$ is $\tau_1\tau_2$-$\ddot{g}$-closed.

Proof:
Suppose $\{x\}$ is not τ_1-sg-closed, then $\{x\}^c$ will not be τ_1-sg-open. $\Rightarrow$ X is the only τ_1-sg-open set containing $\{x\}^c$.$\Rightarrow \tau_2 - cl\{x\}^c \subset X$. $\Rightarrow \{x\}^c$ is $\tau_1\tau_2$-$\ddot{g}$-closed.$\Rightarrow \{x\}$ is $\tau_1\tau_2$-$\ddot{g}$-open.

Theorem 3.19
Arbitrary union of $\tau_1\tau_2$-$\ddot{g}$-closed sets $\{A_i, i \in I\}$ in a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-$\ddot{g}$-closed if the family $\{A_i, i \in I\}$ is locally finite on X.

Proof:
Let $\{A_i / i \in I\}$ be locally finite on X and each A_i is $\tau_1\tau_2$-$\ddot{g}$-closed in X.
To prove: $\cup A_i$ is $\tau_1\tau_2$-$\ddot{g}$-closed.
Let $\cup A_i \subset U$ where U is τ_1-sg-open.$\Rightarrow A_i \subset U$,for every $i \in I$. $\Rightarrow \tau_2 - cl(A_i) \subset U$ for every $i \in I$. $\Rightarrow \cup \tau_2 - clAi \subset U$. Since Ais locally finite,$\tau_2 - cl(\cup A_i) = \cup \tau_2 - cl(A_i)$.$\Rightarrow \tau_2 - cl(\cup A_i) \subset U$ whenever $\cup A_i \subset U$ where U is τ_1-sg-open.$\Rightarrow \cup A_i$ is $\tau_1\tau_2$-$\ddot{g}$-closed.

4. $\tau_1\tau_2$-$\ddot{g}$-open sets

Definition 4.1
A subset A of bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-$\ddot{g}$-open iff $X - A$ is $\tau_1\tau_2$-$\ddot{g}$-closed.

Example 4.2
In example 3.2, $\phi, X, \{a\}, \{a, b\}$ are $\tau_1\tau_2$-$\ddot{g}$-open sets in X.

Theorem 4.3
A set A is $\tau_1\tau_2$-$\ddot{g}$-open iff $F \subseteq \tau_2 - int(A)$ where F is τ_1-sg-closed and $F \subseteq A$.

Proof:
Suppose A is $\tau_1\tau_2$-$\ddot{g}$-open. Then A^c is $\tau_1\tau_2$-$\ddot{g}$-closed. Suppose that F is τ_1-sg-closed and $F \subseteq A$. Then F^c is τ_1-sg-open and $A^c \subseteq F^c$. Therefore $\tau_2 - cl(A^c) \subseteq F^c$ (since A^c is$\tau_1\tau_2$-$\ddot{g}$-closed) .$\Rightarrow X - \tau_2 int(A) \subset F^c$[since $cl(X - A) = X - int(A)$]. Hence $F \subseteq \tau_2 - int(A)$.

Conversely, suppose that $F \subseteq \tau_2 - int(A)$ where F is τ_1-sg-closed and $F \subseteq A$.Then $A^c \subseteq F^c$ and F^c is τ_1-sg-open. Take $U = F^c$. Since $F \subseteq \tau_2 - int(A)$, we have

$[\tau_2 - int(A)]^c \subseteq F^c = U$. $\Rightarrow \tau_2 - cl(A^c) \subseteq U$[since $cl(A^c) = (intA)^c$]. Therefore A^c is $\tau_1\tau_2$-$\ddot{g}$-closed.Thus A is $\tau_1\tau_2$-$\ddot{g}$-open.

Remark 4.4
Every τ_1-open set is $\tau_1\tau_2$-$\ddot{g}$-open but the converse is not true in general as can be seen from the following two examples.

Example 4.5
In example 3.4,
$\tau_1\tau_2$-$\ddot{g}$-open sets $\phi, X, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}$.
$\{a, c\}$ is $\tau_1\tau_2$-$\ddot{g}$-open in X but not τ_1-open in X.

Theorem 4.6
If A and B are $\tau_1\tau_2$-$\ddot{g}$-open sets in bitopological space (X, τ_1, τ_2) then their intersection is a $\tau_1\tau_2$-$\ddot{g}$-open set.

Proof:
If A and B are $\tau_1\tau_2$-$\ddot{g}$-open sets, then A^c and B^c are $\tau_1\tau_2$-$\ddot{g}$-closed sets. $A^c \cup B^c$ is $\tau_1\tau_2$-$\ddot{g}$-closed sets (by theorem 3.11).$(A \cap B)^c$is $\tau_1\tau_2$-$\ddot{g}$-closed.$\Rightarrow A \cap B$ is a$\tau_1\tau_2$-$\ddot{g}$-open set.

Remark 4.7
The union of two $\tau_1\tau_2$-$\ddot{g}$-open sets is need not be $\tau_1\tau_2$-$\ddot{g}$-opens set in X.

This can be seen from the following example.

Example 4.8
Let $X = \{a, b, c, d\}$; $\tau_1 = \{X, \phi, \{a, b\}\}$; $\tau_2 = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$
$\tau_1\tau_2$-$\ddot{g}$-closed set are
$X, \phi, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
$\tau_1\tau_2$-$\ddot{g}$-open set are
$X, \phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

Here {a} and {b} are $\tau_1\tau_2$-$\ddot{g}$-open sets but they union is not $\tau_1\tau_2$-$\ddot{g}$-open set.

Theorem 4.9
The arbitrary intersection of $\tau_1\tau_2$-$\ddot{g}$-open sets $A_i, i \in I$ in a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$-$\ddot{g}$-open if the family $\{A_i^c, i \in I\}$ is locally finite in (X, τ_1).

Proof:
Let $\{A_i^c, i \in I\}$ is locally finite in (X, τ_1) and $A_i$is $\tau_1\tau_2$-$\ddot{g}$-open in X for each $i \in I$. Then A_i^c is $\tau_1\tau_2$-$\ddot{g}$-closed in X for each $i \in I$. By theorem 3.19, we have $\cup(A_i^c)$ is $\tau_1\tau_2$-$\ddot{g}$-closed in X.Consequently, let $(\cap A_i)^c$ is $\tau_1\tau_2$-$\ddot{g}$-closed in X. therefore $\cap A_i$ is $\tau_1\tau_2$-$\ddot{g}$-open in X.

Theorem 4.10
If $\tau_2 - int(A) \subset B \subset A$ and A is $\tau_1\tau_2$-$\ddot{g}$-open in X then B is also $\tau_1\tau_2$-$\ddot{g}$-open in X.

Proof:
Suppose $\tau_2 - int(A) \subset B \subset A$ and A is $\tau_1\tau_2$-$\ddot{g}$-open. Then $A^c \subset B^c \subset X - \tau_2 - int(A) = \tau_2 - cl(X - A) = \tau_2 - cl(A^c)$.Since A^c is $\tau_1\tau_2$-$\ddot{g}$-closed set, by theorem 3.14, B^c is $\tau_1\tau_2$-$\ddot{g}$-closed set. ⇒B is $\tau_1\tau_2$-$\ddot{g}$-open in X.

Theorem 4.11
If a set A is $\tau_1\tau_2$-$\ddot{g}$-closed in X, then $\tau_2 - cl(A) - A$ is $\tau_1\tau_2$-$\ddot{g}$-open set.

Proof:
Suppose that A is $\tau_1\tau_2$-$\ddot{g}$-closed in X. To prove $\tau_2 - cl(A) - A$ is $\tau_1\tau_2$-$\ddot{g}$-open set.Let F be τ_1-sg-closed set and $F \subseteq \tau_2 - cl(A) - A$. Since A is $\tau_1\tau_2$-$\ddot{g}$-closed set in X, we have $\tau_2 - cl(A) - A$ contains no non-empty τ_1-sg-closed set. Since $F \subseteq \tau_2 - cl(A) - A$,
we have
$F = \phi \subseteq \tau_2 - int(\tau_2 - cl(A) - A)$.
Therefore $\tau_2 - cl(A) - A$ is $\tau_1\tau_2$-$\ddot{g}$-open set.

Theorem 4.12
If a set A is $\tau_1\tau_2$-$\ddot{g}$-open in a bitopological space (X,τ_1,τ_2) then $G = X$ whenever G is τ_1-sg-open satisfies $\tau_2 - int(A) \cup A^c \subseteq G$.

Proof:
Suppose that A is $\tau_1\tau_2$-$\ddot{g}$-open in a bitopological space (X,τ_1,τ_2) and G is τ_1-sg-open and $\tau_2 - int(A) \cup A^c \subseteq G$. Then $G^c \subseteq [\tau_2 - int(A) \cup A^c]^c = \tau_2 - cl(A^c) - A^c$. Since G is τ_1-sg-open, we have G^c is τ_1-sg-closed. Since A is $\tau_1\tau_2$-$\ddot{g}$-open, we have A^c is $\tau_1\tau_2$-$\ddot{g}$-closed. Therefore $\tau_2 - cl(A^c) - A^c$ contains no non-empty τ_1-sg-closed set in X. (by theorem 3.17). Consequently $G^c = \phi$. Hence $G = X$.

Example 4.13
Let $X = \{a,b,c,d\}$; $\tau_1 = \{X,\phi,\{a,b\}\}$; $\tau_2 = \{X,\phi,\{b\},\{c\},\{b,c\}\}$
$\tau_1\tau_2$-$\ddot{g}$-open set are
$X,\phi,\{a\},\{b\},\{c\},\{a,c\},\{b,c\},\{a,b,c\}$.

$$A = \{c,d\};\ A^c = \{a,b\}$$

$$\tau_2 - int(A) = \tau_2 - int(\{c,d\}) = \{c\}$$

$\tau_2 - int(A) \cup A^c = \{c\} \cup \{a,b\} = \{a,b,c\} \subseteq X$.
X is τ_1-sg-open but $A = \{c,d\}$ is not $\tau_1\tau_2$-$\ddot{g}$-open.

Theorem 4.14
The intersection of $\tau_1\tau_2$-$\ddot{g}$-open set and τ_2-open set is always $\tau_1\tau_2$-$\ddot{g}$-open set.

Proof:
Suppose that A is $\tau_1\tau_2$-$\ddot{g}$-open set and B is τ_2-open set. Since B is τ_2-open, we have B^c is τ_2-closed."Every τ_2-closed set is $\tau_1\tau_2$-$\ddot{g}$-closed ".Therefore $B^c$is $\tau_1\tau_2$-$\ddot{g}$-closed. ⇒B is $\tau_1\tau_2$-$\ddot{g}$-open. By theorem 4.6, $A \cap B$ is $\tau_1\tau_2$-$\ddot{g}$-open set.

References

[1] Abd El-Monsef M.E, Deeb S.N and Mahmoud R.A, β-open sets and β-continuous mappings, Bull.Fac.Sci.Assiut., 12(1983)77-80.
[2] Bose S, semi open sets, semi continuity and semi open mappings in bitopological spaces, Bull. Cal. Math. Soc., 73(1981) 237-246.
[3] Fukutake T, on generalized closed sets in bitopological spaces, Acta Ciencia of Educ., 35(1985) 19-28.
[4] Fukutake T, semi open sets on bitopological spaces, Bull. Fukuoka. Univ. of Educ 38(1989)1-7.
[5] Indira T, $\tau_1\tau_2$-#g-closed sets in Bitopological spaces, Annals of Pure and Applied mathematics, 7(2) (2014) 27-34.
[6] Kelly J.C, Bitopological spaces, Proc. London. Math. Soc. 13(1963) 71-89.
[7] Levine N, Generalized closed sets in topological spaces, Rend circ Math. Palermo, 19(2) (1970) 89-96.
[8] Levine N, semi open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963) 36-41.
[9] Qays Hatem Imran, Generalized alpha star star closed sets in bitopological spaces, Gen. Math. Note, 22(2) (2014) 93-102.
[10] Ravi O and Ganesan S, $\ddot{g}$-closed sets in topology, International Journal of Computer Science and Emerging Technologies, 2(3)(2011),330-337.
[11] Veerakumar M.K.R.S, #g-closed set in topological spaces, Antarctica J.Math., 2(2)(2005) 234-258