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Euler - Darboux Equation Involving Exp x of Convolution Type-I

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Abstract: ***In this paper, the authors have stablished the solution of Euler-Darboux equation involving exponential functions of convolution type I by using fractional integration. Result is useful and plays important role in physical & mathematical sciences.***

Keywords: Euler-Darboux equation, fractional integral operator, Gauss hypergeometric function, Hölder Continuity, singularity of function

1. Introduction

Recently Authors in paper [1] have defined generalized Lowndes's operator and Hankel operator involving with exponential function. In another paper Authors in paper [2] discussed the solution of certain special quadruple integral equations associated with multi-dimensional inverse Mellin transform of given function by the application of extended form of Erdélyi—Kober operators.

In this paper authors have discussed Euler-Darboux equation associated with exponential function of convolution type-I by following the method due to [6, 7, 8, 9]. In the section 2 definition of the fractional integrals and derivatives are given with several properties which have been used frequently in this paper.

The Euler Darboux equation

$$\frac{\partial^2 u}{\partial x\partial y} - \frac{\beta}{e^x - e^y}\frac{\partial u}{\partial x} + \frac{\alpha}{e^x - e^y}\frac{\partial u}{\partial x} = 0 \ (\alpha > 0,\ \beta > 0,\ \alpha+\beta < 1) \tag{1}$$

implies degenerate exponential equations expressed by the characteristic coordinates. So in section (3), we set a problem for equation (1) with boundary conditions on two characteristics that contain our fractional integral or derivative of order less than some members depending upon α, β, where the problem may be regarded as a generalization of the Goursat problem, i.e. initial value problem for equation (1). Then there will be given an expression of a solution for our problem. To see this the problem will be reduced to a dominant singular integral equation with Cauchy Kernel, where several calculations will be carried out in section (4).

2. Generalized Fractional Integral and Derivative

Let α > 0, and β, η δ and t be real numbers. We shall define a fractional integral in paper [5] of real and continuous function $f(x)$ on (a, ∞) which may have an infinity at x = a of order less than 1 or –β + η + 1 if β < η or β > η, respectively, by

$$I_{ax}^{\alpha,\beta,\eta}\ f \equiv \frac{(\exp x - \exp a)^{-\alpha-\beta}}{\Gamma(\alpha)}\int_a^x (\exp x - \exp t)^{\alpha-1}$$

$$F\left(\alpha+\beta, -\eta;\ \alpha;\ \frac{\exp x - \exp t}{\exp x - \exp a}\right) \tag{2}$$

Where Γ is the gamma function, F means the Gauss hypergeometric function and

$$F(a;b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\ n!}\,z^n$$

in $|z| < 1$ and its analytic continuation into $|\arg z| < \pi$ and

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The expression (2) is a generalization of the fractional integrals of both Riemann - Liouville, Erdelyi-Kober, Hardy and Littlewood i.e.

$$I_{ax}^{\alpha,-\alpha,\eta}\ f = \frac{1}{\Gamma(\alpha)}\int_a^x (\exp x - \exp t)^{\alpha-1} f(t)dt \equiv R_{ax}^{\alpha}\ f \tag{3}$$

And

$$I_{ax}^{\alpha,\,0,\eta}\ f = \frac{(\exp x - \exp a)^{-\alpha-\eta}}{\Gamma(\alpha)}\int_a^x (\exp x - \exp t)^{\alpha-1}$$

$$.\ (\exp t - \exp a)^{\eta}\ f(t)\,dt \equiv E_{ax}^{\alpha,\eta}\ f \tag{4}$$

respectively. In paper [5], for α < 0 a generalized fractional derivative is defined by

$$I_{ax}^{\alpha,\beta,\eta}\ f \equiv \frac{d^n}{dx^n} I_{ax}^{\alpha+n,\beta-n,\eta-n}\ f, \tag{5}$$

If the right hand side has a definite meaning, where it is assumed that 0 < α + η ≤ 1 and n is positive integer. The relation (5) also valid for α > 0.

A fractional integral whose lower limit is a variable x smaller than the constant upper limit b is defined by

$$J_{xb}^{\alpha,\beta,\eta}\ f \equiv I_{0,\,b-x}^{\alpha,\beta,\eta}\ f_1$$

$$\equiv \frac{(\exp b - \exp x)^{-\alpha-\beta}}{\Gamma(\alpha)}\int_x^b (\exp t - \exp x)^{\alpha-1} F\begin{bmatrix} x \\ x \end{bmatrix} f(t)\,dt \tag{6}$$

where

$$F\begin{bmatrix} x \\ x \end{bmatrix} = F\left(\alpha+\beta, -\eta; \alpha; \frac{\exp t - \exp x}{\exp b - \exp x}\right)$$

Here $\alpha > 0$ and $f_1(t) = f(b\text{-}t)$. If $0 < \alpha + n \leq 1$, we shall define a fractional derivative by

$$J_{xb}^{\alpha,\beta,\eta} f \equiv (-1)^n \frac{d^n}{dx^n} J_{xb}^{\alpha+n,\beta-n,\eta-n} f, \qquad (7)$$

where n is a positive integer. Let us write

$$J_{xb}^{\alpha,-\alpha,\eta} \equiv L_{xb}^{\alpha} f.$$

As found in paper [5], the following product rules hold:

$$I_{ax}^{\alpha,\beta,\eta} I_{ax}^{\gamma,\delta,\alpha+\eta} = I_{ax}^{\alpha+\gamma,\beta+\delta,\eta} \qquad (\alpha,\gamma>0) \qquad (8)$$

$$I_{ax}^{\alpha,\beta,\eta} I_{ax}^{\gamma,\delta,\eta-\beta-\gamma-\delta} = I_{ax}^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta} \qquad (\alpha,\gamma>0) \qquad (9)$$

These are still valid for negative non-integer α.

It is easily seen that $I_{ax}^{0,0,\eta}$ is the identity operator for any h, then the inverse operator of $I_{ax}^{\alpha,\beta,\eta}$ is given by :

$$\left(I_{ax}^{\alpha,\beta,\eta}\right)^{-1} = I_{ax}^{-\alpha,-\beta,\alpha+\eta} \qquad (10)$$

which follow from formulas (8) or (9)

The formulas (8) to (10) hold true also for the operator J defined by equation (6).

3. The Euler-Darboux Equation

Consider the equation (1) in the domain $\Omega=(0<x<y<1)$. It is well known in paper [10] and Book [11] that the solution of equation (1) under the conditions.

$$u\big|_{y=x} = \tau(x), (\exp y - \exp x)^{\alpha+\beta}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right)\Big|_{y=x} = \nu(x) \qquad (11)$$

is expressed as :

$$u(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \tau[\exp x + (\exp y - \exp x)\exp t]$$
$$.(\exp t)^{\beta-1}(1-\exp t)^{\alpha-1}\,dt + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)\Gamma(1-\beta)}(\exp y - \exp x)^{1-\alpha-\beta}$$
$$\int_0^1 \nu[\exp x + (\exp y - \exp x)\exp t](\exp t)^{-\alpha}(1-\exp t)^{-\beta}\,dt \qquad (12)$$

Then the values of u on two characteristics x=0 and y=1 can be written respectively in terms of the fractional integrals (2) and (6) as follows:

$$u^{(1)}(y) \equiv u(0,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{0y}^{\alpha,0,\beta-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)}$$
$$I_{0y}^{1-\beta,\alpha+\beta-1,\beta-1}\nu, 0<y<1, \qquad (13)$$

$$u^{(2)}(x) \equiv u(x,1) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} J_{x1}^{\beta,0,\alpha-1}\tau$$
$$+ \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)} J_{x1}^{1-\alpha,\alpha+\beta-1,\alpha-1}\nu, \qquad 0<x<1 \qquad (14)$$

Now we shall set a problem which is the main subject for the rest of this work.

Let $H^k(T)$ be a class of functions which are defined on a real interval T and Hölder continuous in T with the Hölder index k. Let us denote the open interval (0, 1) by U and its closure by $\overline{U}$.

Problem A : To seek for a solution u(x, y) of equation (1) in Ω such that

(i) $\tau(x) \in H^{k_1}(\overline{U})$ and $\nu(x) \in H^{k_2}(U)$ for some k_1, and k_2 $(0 < k_1, k_2 < 1)$
where $\nu(x)$ may have infinites of order not greater than $1 - \alpha - \beta$ at the end point of U.

(ii) $u^{(1)}(y)$ and $u^{(2)}(x)$ in (13) and (14) satisfy respectively the boundary conditions

$$I_{0y}^{a,b,-a+\beta-1} u^{(1)} = \varphi_1(y), y \in u \quad \text{and} \qquad (15)$$

$$J_{x1}^{c,d,-c+\alpha-1} u^{(2)} = \varphi_2(x), x \in U, \qquad (16)$$

where a, b, c and d are constants such that $\max(-\alpha,\beta-1) < a < \min(\beta, 1-\alpha)$, $b > -\alpha-\beta$, $\max(-\beta, a-1) < c < \min(\alpha, 1-\beta)$ and $d > -\alpha-\beta$, and $\varphi_1 \in H^{k_3}(\overline{U})$ $[\max(a-\beta+1, a+\alpha, -b) < k_3 < 1, \varphi_1(0)=0]$ and $\varphi_2 \in H^{k_4}(\overline{U})$ $[\max(c-\alpha+1, c+\beta, -d-\alpha-\beta+1) < k_4 < 1, \varphi_2(1) = 0]$ are given functions.

Note that if a = b = c = d = 0, problem A is reduced to the Goursat problem. If we note (13), (14) and (8), and replace y by x, the conditions (15) and (16) may be read as :

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{ox}^{a+\alpha,b,-a+\beta-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)}$$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{ox}^{a+\alpha,b,-a+\beta-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)}$$
$$I_{0x}^{a-\beta+1,b+\alpha+\beta-1,-a+\beta-1}\nu = \varphi_1(x) \qquad (17)$$

and

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(a)} I_{x1}^{c+\beta,d,-c+\alpha-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)}$$
$$J_{x1}^{c-\alpha+1,d+\alpha+\beta-1,-c+\alpha-1}\nu = \varphi_2(x) \text{ on } U \qquad (18)$$

thus problem (A) is reduced to

Problem (B): To seek for solutions $\tau(x)$ and $\nu(x)$ of (17) and (18) satisfying the condition (i) in problem A, where $\varphi_1(x)$, $\varphi_2(x)$, a, b, c and d are as given in problem A.

Operating $\left(I_{ox}^{a+\alpha,b,-a+\beta-1}\right)^{-1} = I_{ox}^{-a-\alpha,-b,\alpha+\beta-1}$ on both sides of equation (17), we have

$$\tau(x) + \frac{\Gamma(\beta)\Gamma(1-\alpha-\beta)}{2\Gamma(\alpha+\beta)\Gamma(1-\alpha)} R_{ox}^{1-\alpha-\beta}\nu = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$
$$I_{ox}^{-a-\alpha,-b,\alpha+\beta-1}\varphi_1 \qquad (19)$$

Similarly, by using $\left(J_{ox}^{c+\beta,d,-c+\alpha-1}\right)^{-1} = J_{ox}^{-c-\beta,-d,\alpha+\beta-1}$ we have

$$\tau(x)+\frac{\Gamma(\alpha)\Gamma(1-\alpha-\beta)}{2\Gamma(\alpha+\beta)\Gamma(1-\alpha)}L_{x1}^{1-\alpha-\beta}\nu=\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}J_{x1}^{-c-\beta,-d,\alpha+\beta-1}\varphi_2 \quad (20)$$

The subtraction of equation (20) from equation (19) and the operation of $R_{0x}^{\alpha+\beta-1}$ on both sides given the relation,

$$\nu(x)-\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{2\Gamma(\beta)\Gamma(1-\beta)}R_{0x}^{\alpha+\beta-1}L_{x1}^{1-\alpha-\beta}\ \nu=\Phi_0(x) \quad (21)$$

where,

$$\Phi_0(x)=\frac{2\Gamma(1-\alpha)}{\Gamma(1-\alpha-\beta)}R_{0x}^{\alpha+\beta-1}I_{0x}^{-a-\alpha,-b,\alpha+\beta-1}\varphi_1-\frac{2\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\beta)\Gamma(1-\alpha-\beta)}R_{0x}^{\alpha+\beta-1}J_{x1}^{-c-\beta,-d,\alpha+\beta-1}\varphi_2 \quad (22)$$

It is well known in paper [10] and Book [11], the relation

$$R_{0x}^{-\alpha}L_{x1}^{\alpha}\varphi=\cos\pi\alpha\varphi(x)+\frac{\sin\pi\alpha}{\pi}\int_0^1\left(\frac{\exp u}{\exp x}\right)^{\alpha}\frac{1}{\exp u-\exp x}\varphi(u)du$$

hold valid for $0<\alpha<1$, where the integral is taken in the sense of the Cauchy principal value. Then equation (21) may be written in the form

$$\nu(x)-\frac{\tan\pi\beta}{\pi}\int_0^1\left(\frac{\exp u}{\exp x}\right)^{1-\alpha-\beta}\frac{1}{\exp u-\exp x}\nu(u)du=\frac{\sin\pi\alpha}{\cos\pi\alpha\sin\pi(\alpha+\beta)}\Phi_0(x) \quad (23)$$

Here we have used the relation $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, due to Book [4].

If we use a new unknown function $\mu(x)=\nu(x)\,x^{1-\alpha-\beta}$, then we have from equation (23) the dominant singular integral equation for $\mu(x)$.

$$\mu(x)-\frac{\tan\pi a}{\pi}\int_0^1\frac{\mu(u)}{\exp u-\exp x}du=\Phi(x)\,\text{on}\,u, \quad (24)$$

where

$$\Phi(x)=\frac{2\Gamma(\alpha+\beta)}{\Gamma(\alpha)(\cos\pi\beta)}(\exp x)^{1-\alpha-\beta}R_{0x}^{\alpha+\beta-1}I_{0x}^{-a-\alpha,-b,\alpha+\beta-1}\varphi_1-\frac{2\Gamma(\alpha+\beta)}{\Gamma(\beta)(\cos\pi\beta)}(\exp x)^{-1-\alpha-\beta}R_{0x}^{\alpha+\beta-1}.J_{x1}^{-c-\beta,-d,\alpha+\beta-1}\varphi_2 \quad (25)$$

Now let us solve equation (24) by applying the theory in Gakhov [3]. To this end we shall summarize results of the theory in an applicable form to our equation (24). Let $L=t_0t_1$ be on open non intersecting smooth curve in the complex plane. Functions a(t), b(t) and $f(t)$ are assumed to be Hölder continuous on the closure of L with $a^2(t)-b^2(t)=1$. Then a solution of the dominate singular integral equation.

$$a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-\exp t}d\tau=f(t)\,\text{on}\,L$$

is given by

$$\varphi(t)=a(t)f(t)-\frac{b(t)Z(t)}{\pi i}\int_L\frac{f(\tau)}{Z(\tau)}\frac{d\tau}{(\tau-\exp t)}$$

where

$$Z(t)=\exp\left\{\frac{1}{2\pi i}\int_L\frac{1}{\tau-\exp t}\operatorname{Log}G(\tau)d\tau\right\}$$

and $$G(t)=\frac{a(t)-b(t)}{a(t)+b(t)}$$

The above results are introduced under such assumptions that the index k of the equation is equal to zero and the solution $\varphi(t)$ is sought to be Hölder continuous on L, bounded at $t=t_0$ and unbounded but having integrable order at $t=t_1$, where, if we set $G(t_0)=\rho\exp(i\theta)$ and $G(t_1)=\rho'\exp(i(\theta+\Delta))$, θ and k are chosen such as $-2\pi<\theta\leq 0$ and $k=[(\theta+\Delta)/2\pi]+1$. Here Δ is the change of arg G(t) on L and [x] denotes the greatest integer not exceeding x.

Returning to our equation (24), we have

$$G(x)=\frac{1+i\tan\pi\beta}{1-i\tan\pi\beta}=e^{2\pi i(\beta-1)}$$

Then $\theta=2\pi(\beta-1)$, $\Delta=0$ and $k=0$ are obtained. Therefore the solution of equation (24) is written in the form

$$\mu(x)=\frac{1}{1+\tan^2\pi\beta}\left[\Phi(x)+\frac{\tan\pi\beta}{\pi}\left(\frac{\exp x}{1-\exp x}\right)^{1-\beta}\int_0^1\left(\frac{\tau-\exp u}{\exp u}\right)^{1-\beta}\frac{1}{\exp u-\exp x}\Phi(u)du\right]$$

Which is Hölder continuous in U, bounded at x = 0 and unbounded but having integrable singularity at x = 1. Here we have taken into consideration that $\Phi(x)$ is Hölder continuous on $\overline{U}$, whose continuity will be proved in the next section.

From above we may easily find $\nu(x)$ satisfying the condition (i). Since it will be seen in the next section that $\nu(x)$ is of order greater than $\alpha+\beta-1$ at x = 1. Then in view of equations (19) or (20) $\tau(x)$ satisfying (i) is determined.

4. Regularity of $\Phi(x)$ and singularity of $\nu(x)$

We need the following Lemmas.

Lemma 1: Let a < b and 0 < k < 1. If $\varphi,\psi\in H^k[a,b]$ with $\varphi(a)=\psi(b)=0$, $0>\alpha>-k$, $\beta<k$ and $\eta>\beta-1$, then $I_{ax}^{\alpha,\beta,\eta}\varphi, J_{xb}^{\alpha,\beta,\eta}\psi\in H^{k+\alpha}[a,b]$.

Proof:
As in easily calculated, we have

$$I_{ax}^{\alpha,\beta,\eta}\varphi = \frac{d}{dx} I_{ax}^{\alpha+1,\beta-1,\eta-1}\varphi = \frac{\Gamma(-\beta+\eta+1)}{\Gamma(-\beta+1)\Gamma(\alpha+\eta+1)}$$

$$(\exp x - \exp a)^{-\beta}\varphi(x) + \frac{(\exp x-\exp a)^{-\alpha-\beta}}{\Gamma(\alpha)}\int_a^x (\exp x - \exp t)^{\alpha-1}$$

$$F\left(\alpha+\beta,\eta;\alpha;\frac{\exp x-\exp t}{\exp x-\exp a}\right)\{\varphi(t)-\varphi(x)\}\,dt \quad (26)$$

It is well known in Book [4] that the hypergeometric function F(a,b;c;z) has orders such that 0(1), (z→0); $0[(1-z)^{\min(0,\, c-a-b)}]$, (z→1), then the kernel of integral in equation (26) has the properties :

$$(\exp x-\exp t)^{\alpha-1}\ F\left(\alpha+\beta,\eta;\alpha;\frac{\exp x-\exp t}{\exp x-\exp a}\right)$$

$$= 0[(\exp x-\exp t)^{\alpha-1}](t\to x)$$

and $= 0[(\exp t-\exp a)^{\min(0,-\beta+\eta)}],(t\to a)$

Then in order to prove the Hölder continuity of $I_{ax}^{\alpha,\beta,\eta}\varphi$ it is enough to show that

$$A(x) = \int_a^x (\exp t-\exp a)^q(\exp x-\exp t)^{\alpha-1}$$

$$\{\varphi(t)-\varphi(x)\}\,dt \in H^{k+\alpha}[a,b] \quad (27)$$

where q = min (0, – β + η). Let a ≤ x < x + h ≤ b and consider the difference

$$A(x+h)-A(x) = \int_a^{x+h}(\exp t-\exp a)^q(\exp x+\exp h-\exp t)^{\alpha-1}$$

$$[\varphi(t)-\varphi(x+h)]dt - \int_a^x(\exp t-\exp a)^q.(\exp x-\exp t)^{\alpha-1}[\varphi(t)-\varphi(x)]dt$$

$$= \int_a^x(\exp t-\exp a)^q[(\exp x+\exp h-\exp t)^{\alpha-1}-(\exp x-\exp t)^{\alpha-1}]$$

$$[\varphi(t)-\varphi(x)]dt + \int_a^x(\exp t-\exp a)^q(\exp x+\exp h-\exp t)^{\alpha-1}dt$$

$$+\int_x^{x+h}(\exp x+\exp h-\exp t)^{\alpha-1}[\varphi(t)-\varphi(x+h)]dt = I_1+I_2+I_3$$

$$(28)$$

Changing the variables of integration suitably, we have as h→0

$$I_1 = 0\left[\int_a^x(\exp t-\exp a)^q(\exp x-\exp t)^k\right.$$

$$\left.\left\{(\exp x+\exp h-\exp t)^{\alpha-1}-(\exp x-\exp t)^{\alpha-1}\right\}dt\right]$$

$$=0\left[(\exp h)^{k+\alpha}(\exp x-\exp a+\exp h)^q\right.$$

$$\left.\int_1^{1+\frac{x-a}{h}}(\upsilon-1)^k\{\upsilon^{\alpha-1}-(\upsilon-1)^{\alpha-1}\}\right.$$

$$\left.\left(1-\frac{\exp h}{\exp x-\exp a+\exp h}\upsilon\right)^q d\upsilon\right]$$

$$= 0\,(\exp h)^{k+\alpha} \quad (29)$$

$$I_2 = 0\left[(\exp x-\exp a)^{q+1}(\exp h)^k(\exp x-\exp a+\exp h)^{\alpha-1}\right.$$

$$\left.\int_0^1\upsilon^q\left(1-\frac{\exp x-\exp a}{\exp x-\exp a+\exp h}\upsilon\right)^{\alpha-1}d\upsilon\right]$$

$$=0\left[(\exp h)^k F\left(q+1,1-\alpha;q+2;\frac{\exp x-\exp a}{\exp x-\exp a+\exp h}\right)\right]$$

$$= 0\,(\exp h)^{k+\alpha} \quad (30)$$

$$I_3 = 0\left[\int_x^{x+h}(\exp t-\exp a)^q(\exp x+\exp h-\exp t)^{k+\alpha-1}dt\right]$$

$$=0\left[(\exp h)^{h+\alpha}(\exp x-\exp a+\exp h)^q\int_0^1\upsilon^{k+\alpha-1}\right.$$

$$\left.\left(1-\frac{\exp h}{\exp x-\exp a+\exp h}\upsilon\right)^q d\upsilon\right]$$

$$=0\left[(\exp h)^{k+\alpha}F\left(k+\alpha,-q;k+\alpha+1;\frac{\exp h}{\exp x-\exp a+\exp h}\right)\right]$$

$$= 0\left[(\exp h)^{k+\alpha}\right] \quad (31)$$

Since k + α > 0, q > – 1 and

$$\int_0^1(\exp t)^{a-1}(1-\exp t)^{c-\alpha-1}\{1-(\exp z)(\exp t)\}^{-b}dt$$

$$=\Gamma(a)\frac{\Gamma(c-a)}{\Gamma(c)}\Gamma(a,b;c;\exp z)$$

[Re c > Re a > 0, |arg (1-z)| < π, z ≠ 1] in paper [4]. When equations (28) to (31) are used, equation (27) can be reading proved, proof of $J_{xb}^{q,\beta,\eta}\Psi\in H^{k+\alpha}[a,b]$ are parallel.

Lemma 2: Under the condition of Lemma 1 except φ(a) ≡ Ψ(b) = 0 and β < k, $(x-\alpha)^{\beta}\ I_{ax}^{\alpha,\beta,\eta}\varphi$ and $(b-x)^{\beta}\ J_{xb}^{\alpha,\beta,\eta}\psi$ belong to H^{k+a} [a, b].

Proof : The first results immediate by noting the relation (26) and the proof of Lemma 1. The second is similar. Now let us show that Φ(x) in (25) is Hölder continuous on $\overline{U}$. From Lemma 1 we can easily find that $g(x)\equiv I_{ax}^{-a-\alpha,-b,\alpha+\beta-1}\varphi,\in H^{k_3-a-\alpha}(\overline{U})$, since $k_3 > a + \alpha$, $k_3 + b > 0$, b + α + β > 0 and $\varphi_1(0) = 0$. Then Lemma 2 implies $x^{1-\alpha-\beta}R_{ox}^{\alpha+\beta-1}g\in H^{k_3-a+\beta-1}(\overline{U})$. Similarly Lemma 1 guarantees that h(x) $\equiv J_{x1}^{-c-\beta,-d,\alpha+\beta-1}\varphi_2\in H^{k_4-c-\beta}(\overline{U})$ by virtue of the assumptions $k_4 > c + \beta$, $k_4 + d > 0$, $d + \alpha + \beta > 0$ and $\varphi_2(1) = 0$. Thus from Lemma 2 $x^{1-\alpha-\beta} R_{0x}^{\alpha+\beta-1} h \in H^{k_4-c+d-1}(\bar{U})$. Therefore it has been proved that $\Phi(x) \in H^{\min(K_3-a+\beta-1, K_4-c+\alpha-1)}(\bar{U})$.

To see the order of singularity of $\nu(x)$ at $x = 1$ we operate $(J_{x1}^{c-\alpha+1,d+\alpha+\beta-1,-c+\alpha-1})^{-1} = (J_{x1}^{-c-\alpha-1,-d-\alpha-\beta+1,0})$ on both sides of equation (18). Then we have

$$\nu(x) = \frac{2\Gamma(\alpha+\beta)\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} L_{x1}^{\alpha+\beta-1} \tau + \frac{2\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta)} J_{x1}^{-c+\alpha-1,-d-\alpha-\beta+1,0} \varphi_2 \qquad (32)$$

Thus Lemma 1 implies that $\nu(x) = O[(1-x)^{\min(k_1+\alpha+\beta-1, k_4-c+\alpha-1}](x \to 1)$ where k_1 is the Hölder index of τ. Here the condition $\tau(1) = 0$ has been used, which following by virtue of equation (20) and Lemma 1. Hence we have established that ν is of order greater than $\alpha + \beta - 1$ at $x = 1$.

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