ξ-Normal and ξ-Regular Spaces in Topological Spaces

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Abstract: ***The aim of this paper is to introduce and study two new classes of spaces, namely ξ-normal and ξ-regular spaces in topological spaces. The relationships among normal, p-normal, α-normal, β-normal and ξ-normal spaces are investigated. Moreover, we introduced some functions such as gξ-closed, ξ-gξ-closed, pre ξ-open. We obtained some characterizations of ξ-normal and ξ-regular spaces, properties of the forms of gξ-closed functions and preservation theorems for ξ-normal and ξ-regular spaces.***

Keywords: ξ-closed sets, ξ-normal, ξ-regular spaces, gξ-closed and ξ-gξ-closed functions

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1. Introduction

Levine [**3**] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. α-open sets were introduced by Njastad [**7**]. Devi et al. [**2**] introduced the concept of ξ-closed sets. Nour [**8**] introduced the notion of p-normal spaces and obtained their characterizations and preservation theorems. Paul and Bhattacharyya [**9**] obtained some properties of p-normal spaces. Benchalli et al. [**1**] introduced the notion of α-normal spaces and obtained their characterizations and preservation theorems. Mahmoud et al. [**4**] introduced the notion of β-normal spaces and obtained their characterizations and preservation theorems. Recently, Sharma et al. [**10**] introduced a new class of regular spaces called ξ-regular spaces by using ξ-open sets introduced by Devi et al. [**2**] and obtained several properties such as characterizations and preservation theorems for ξ-regular spaces.

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by **Cl(A)** and **Int(A)** respectively. A is said to be **α-open** [**1**] if $A \subset Int(Cl(Int(A)))$. The complement of a α-open set is said to be **α-closed** [**1**]. The intersection of all α-closed sets containing A is called **α-closure** [2] of A, and is denoted by **αCl(A)**.

2.1 Definition. A subset A of a space (X, τ) is said to be

1. generalized closed (briefly **g-closed**) **[3]** if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

2. generalized α-closed (briefly **αg-closed**) [**6**]) if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

3. generalized α-closed (briefly **gα-closed**) [**5**]) if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is α-open in X.

4. **ξ-closed** **[2]** if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is gα-open in X.

5. g-open (resp**. αg-open, gα-open**, **ξ-open**) if the complement of A is g-closed (resp. αg-closed, gα-closed, ξ-closed).

The intersection of all ξ-closed sets containing A is called **ξ-closure** of A, and is denoted by **ξCl(A)**. The **ξ-interior** of A, denoted by **ξInt(A)**, is defined as union of all ξ-open sets contained in A. The family of all ξ-closed (resp. ξ-open) sets of a space X is denoted by ξC(X) (resp. ξO(X)).

2.2 Lemma. Let A be a subset of a space X and $x \in X$. The following properties hold for ξCl(A):

(i) $x \in \xi C1(A)$ if and only if $A \cap U \neq \phi$ for every $U \in \xi O(X)$ containing x.
(ii) A is ξ-closed if and only if A = ξCl(A).
(iii) $\xi C1(A) \subset \xi C1(B)$ if $A \subset B$.
(iv) ξC1(ξC1(A)) = ξC1(A).
(v) ξC1(A) is ξ-closed.

2.3 Definition. A subset A of a space X is said to be **generalized ξ-closed** (briefly **gξ-closed**) if $\xi Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

2.4 Remark. We have the following implications for the properties of subsets:

$$\begin{array}{ccc} \text{closed} & \Rightarrow & \text{g-closed} \\ \Downarrow & & \Downarrow \\ \alpha\text{-closed} & \Rightarrow & \alpha\text{g-closed} \\ \Downarrow & & \Downarrow \\ \xi\text{-closed} & \Rightarrow & \text{g}\xi\text{-closed} \end{array}$$

Where none of the implications is reversible as can be seen from the following examples:

2.5 Example Let X = {a, b, c} and τ = {∅, {a}, X}. Then A= {b} is g-closed but not closed.

2.6 Example. Let X = {a, b, c,} and $\tau = \{\varnothing, \{a\}, X\}$. Then A = {a, b} is g-closed as well as gξ-closed.

2.7 Example Let X = {a, b, c, d} and $\tau = \{\varnothing, \{c\},\{d\}, \{a, c\}, \{c, d\},\{a, c, d\}, X\}$ Then A ={a} is α-closed as well as ξ-closed but not closed.

2.8 Example Let X = {a, b, c, d} and $\tau = \{\varnothing, \{c, d\}, X\}$. Then A = {a, b, c} is ξ-closed. But it is neither α-closed nor closed.

2.9 Lemma. A subset A of a space X is gξ-open in X if and only if $F \subset$ ξInt(A) whenever $F \subset$ A and F is closed in X.

3. Generalized ξ-closed functions

3.1 Definition. A function f: X → Y is said to be **ξ-closed** [**2**] if for each closed set F of X, f(F) is ξ-closed in Y**.**

3.2 Definition. A function f: X → Y is said to be

(i) **generalized ξ-closed** (briefly **gξ-closed**) if for each closed set F of X, f (F) is gξ-closed in Y.

(ii) **ξ-generalized ξ-closed** (briefly **ξ-gξ-closed**) if for each ξ-closed set F of X, f (F) is gξ-closed in Y.

3.3 Remark. Every closed function is ξ-closed but not conversely. Also, every ξ-closed function is gξ-closed because every ξ-cosed set is gξ-closed. It is obvious that both ξ-closedness and ξ-gξ-closedness imply gξ-closedness.

3.4 Theorem. A surjective function f: X → Y is gξ-closed (resp. ξ-gξ-closed) if and only if for each subset B of Y and each open (resp. ξ-open) set U of X containing $f^{-1}(B)$, there exists a gξ-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose that f is gξ-closed (resp. ξ-gξ-closed). Let B be any subset of Y and U be open (resp. ξ-open) set of X containing $f^{-1}(B)$. Put V = Y – f(X – U). Then the complement V^c of V is V^c = Y – V = f(X – U). Since X – U is closed in X and f is gξ-closed, f(X – U) = V^c is gξ-closed. Therefore, V is gξ-open in Y. It is easy to see that $B \subset V$ and $f^{-1}(V) \subset U$.

Conversely, let F be a closed (resp. ξ-closed) set of X. Put B = Y – f(F), then we have $f^{-1}(B) \subset X - F$ and X – F is open (resp. ξ-open) in X. Then by assumption, there exists a gξ-open set V of Y such that B = Y – f(F) $\subset$ V and $f^{-1}(V) \subset X - F$. Now $f^{-1}(V) \subset X - F$ implies $V \subset Y - f(F)$ = B. Also $B \subset V$ and so B =V. Therefore, we obtain f(F) = Y – V and hence f (F) is gξ-closed in Y. This shows that f is gξ-closed (resp. ξ-gξ-closed).

3.5 Remark. We can prove the necessity part of the above theorem by replacing each set to closed set in the form of the proposition given below:

3.6 Proposition. If a surjective function f: X → Y is gξ-closed (resp. ξ-gξ-closed) then for a closed set F of Y and for any open (resp. ξ-open) set U of X containing $f^{-1}(F)$, there exists a ξ-open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$.

Proof. By **Theorem 3.4**, there exists a gξ-open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since F is closed, by **Lemma 2.9** we have $F \subset$ ξInt (W). Put V = ξInt(W). Then $V \in$ ξO(Y), $F \subset V$ and $f^{-1}(V) \subset U$.

3.7 Proposition. If f: X → Y is continuous ξ-gξ-closed and A is gξ-closed in X, then f(A) is gξ-closed in Y.

Proof. Let V be a open set of Y containing f(A). Then $A \subset f^{-1}(V)$. Since f is continuous, $f^{-1}(V)$ is open in X. Since A is gξ-closed in X, by a definition, we get $\xi C1(A) \subset f^{-1}(V)$ and hence $f(\xi C1(A)) \subset V$. Since f is ξ-gξ-closed and ξC1(A) is ξ-closed in X, f(ξC1(A)) is gξ-closed in Y and hence we have $\xi C1(f(\xi C1(A))) \subset V$. By definition of the ξ-closure of a set, $A \subset \xi C1(A)$ which implies $f(A) \subset f(\xi C1(A))$ and using **Lemma 2.2,** $\xi C1(f(A)) \subset \xi C1(f(\xi C1(A))) \subset U$. That is $\xi C1(f(A)) \subset U$. This shows that f (A) is gξ-closed in Y.

3.8 Definition. A function f**:** X → Y is said to be **ξ-irresolute** [**2**] if for each $V \in$ ξO(Y), $f^{-1}(V) \in$ ξO(X).

3.9 Proposition. If f: X → Y is an open ξ-irresolute bijection and B is gξ-closed in Y, then $f^{-1}(B)$ is gξ-closed in X.

Proof. Let U be a open set of X containing $f^{-1}(B)$. Then $B \subset f(U)$ and f(U) is open in Y. Since B is gξ-closed in Y, $\xi C1(B) \subset f(U)$ and hence we have $f^{-1}(\xi C1(B)) \subset U$. Since f is ξ-irresolute, $f^{-1}(\xi C1(B))$ is ξ-closed in X (Theorem 2.1 (i) and (v)), we have $\xi C1(f^{-1}(B)) \subset f^{-1}(\xi C1(B) \subset U$. This shows that $f^{-1}(B)$ is gξ-closed in X.

3.10 Theorem. Let f: X → Y and h: Y → Z be the two functions, then

(i) If hof: X → Z is gξ-closed and if f: X → Y is a continuous surjection, then h: X → Z is gξ-closed.
(ii) If f: X → Y is gξ-closed with h: Y → Z is continuous and ξ-gξ-closed, then hof: X → Z is gξ-closed.
(iii) If f: X → Y is closed and h: Y → Z is gξ-closed, then hof: X → Z is gξ-closed.

Proof.

(i) Let F be a closed set of Y. Then $f^{-1}(F)$ is closed in X since f is continuous. By hypothesis (hof) $(f^{-1}(F))$ is gξ-closed in Z. Hence h is gξ-closed.
(ii)The proof follows from the P**roposition 3.7**.
(iii)The proof is obvious from definitions.

4. ξ-Normal spaces

4.1 Definition. A space X is said to be **ξ-normal** (resp. **α-normal** [**1**], **p-normal** [**8**, **9**], **β-normal** [**4**]) if for any pair of disjoint closed sets A, B of X, there exist disjoint ξ-open (resp. α-open, p-open, β-open) sets U and V such that $A \subset U$ and $B \subset V$.

By the definitions stated above, we have the following diagram:

normality ⇒ α-normalily ⇒ p-normality ⇒ β-normality

⇓

ξ-normalily

Where none of the implications is reversible as can be seen from the following examples:

4.2 Example. Let X = {a, b, c, d} and = {ϕ, {b}, {d}, {b, d}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Taking β-open sets, U = {a, b} and V = {c, d} such that $A \subset U$ and $B \subset V$. Hence the space X is β-normal. But the space X is neither p-normal nor α-normal, since the sets U and V are neither p-open nor α-open..

4.3 Example. Let X = {a, b, c, d} and τ = {∅, {b, d}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Taking p-open sets, U = {a, b} and V = {c, d} such that $A \subset U$ and $B \subset V$. Hence the space X is p-normal as well as β-normal, since every p-open sets are β-open. But the space X is neither normal nor α-normal, since the sets U and V are neither open nor α-open.

4.4 Theorem. The following properties are equivalent for a space X:

(a) X is ξ-normal.
(b) For each pair of disjoint closed sets A, B of X, there exist disjoint gξ-open sets U and V such that $A \subset U$ and $B \subset V$.
(c) For each closed set A and any open set V containing A, there exists a gξ-open set U such that $A \subset U \subset \xi C1(U) \subset V$.
(d) For each closed set A and any g-open set B containing A, there exists a gξ-open set U such that $A \subset U \subset \xi C1(U) \subset Int(B)$.
(e) For each closed set A and any g-open set B containing A, there exists a ξ-open set set G such that $A \subset G \subset \xi C1)G) \subset Int (B)$.
(f) For each g-closed set A and any open set B containing A, there exists a ξ-open set U such that $C1(A) \subset U \subset \xi C1(U) \subset B$.
(g) For each g-closed set A and any open set B containing A, there exists a gξ-open set G such that $C1(A) \subset G \subset \xi C1(G) \subset B$

Proof. (a) ⇒ (b). This proof is obvious since every ξ-open set is gξ-open.

(b) ⇒ (c). Let A be a closed set and let V be an open set containing A. Since A and X – V are disjoint closed sets of X, there exist gξ-open sets U and W of X such that $A \subset U$ and $X - V \subset W$ and $U \cap W = \phi$. By **Lemma 2.9,** we get $X - V \subset \xi Int(W)$. Since $U \cap \xi Int(W) = \phi$, we have $\xi C1(U) \cap \xi Int(W) = \phi$ and hence $\xi C1(U) \subset X - \xi Int(W) \subset V$. Therefore, we obtain $A \subset U \subset \xi C1 \subset V$.

(c) ⇒ (a). Let A and B be the disjoint closed sets of X. Since X – B is an open set containing A, there exists a gξ-open set G such that $A \subset G \subset \xi C1(G) \subset X - B$. By **Lemma 2.9**, we have $A \subset \xi Int(G)$. Put $U = \xi Int(G)$ and $V = X - \xi C1(G)$. Then U and V are disjoint ξ-open sets such that $A \subset U$ and $B \subset V$. Therefore X is ξ-normal.

Since every ξ-open set is gξ-open and every closed (resp. open) set is g-closed (resp. g-open), it is obvious that (d) ⇒ (e) ⇒ (c) and (f) ⇒ (g) ⇒ (c).

(c) ⇒ (e). Let A be a closed set of X and let B be a g-open set such that $A \subset B$. Since B is g-open and A is closed, $A \subset Int(B)$ by **Lemma 2.9**. Therefore by (c), there exists a gξ-open set U such that $A \subset U \subset \xi C1(U) \subset \xi Int(B)$.

(e) ⇒ (d). Let A be a closed set of X and let B be a g-open set such that $A \subset B$. Then there exists a gξ-open set G such that $A \subset G \subset \xi C1(G) \subset Int(B)$ by **Lemma 2.9**. Since G is gξ-open, $A \subset \xi Int(G)$. Put $U = \xi Int(G)$, then U is ξ-open and $A \subset U \subset \xi Cl(U) \subset Int(B)$.

(c) ⇒ (g). Let A be a g-closed set of X and let B be an open set such that $A \subset B$. Then $Cl(A) \subset B$. Therefore by (c), there exists a gξ-open set U Such that $Cl(A) \subset U \subset \xi Cl(U) \subset B$.

(g) ⇒ (f). Let A be a g-closed set of X and let B be an open set such that $A \subset B$. Then there exist a gξ-open set G such that $Cl(A) \subset G \subset \xi Cl(G) \subset B$. Since G is gξ-open and the closed set $Cl(A) \subset G$, we have $Cl(A) \subset \xi Int(G)$ by **Lemma 2.9.** Put $U = \xi Int(G)$. Then, U is ξ-open and $Cl(A) \subset U \subset \xi Cl(U) \subset B$.

4.5 Theorem. If $f: X \to Y$ is continuous gξ-closed surjection and X is normal, then Y is ξ-normal.

Proof. Let A and B be the disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X since f is continuous. Since X is normal, there exists disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By **Proposition 3.6**, there exist gξ-open sets G and H of Y such that $A \subset G$, $B \subset H$ and $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \phi$ and hence $G \cap H = \phi$. It follows from **Theorem 4.4** that Y is ξ-normal.

4.6 Theorem. If $f: X \to Y$ is continuous ξ-gξ-closed surjection and X is ξ-normal, then Y is ξ-normal.

Proof. Let A and B the disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X. Since X is ξ-normal, there exist disjoint ξ-open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since f is ξ-gξ-closed, by **Proposition 3.6**, there exist ξ-open sets G and H of Y such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Since U and V are disjoint, we have $G \cap H = \phi$. This shows that Y is ξ-normal.

5. ξ-Regular Spaces

5.1 Definition. A space X is said to be **ξ-regular** [**10**] (resp. **α-regular** [**1**]) if for each closed set F of X, and each point $x \in X - F$, there exist disjoint ξ-open (resp. α-open) set U, V such that $F \subset U$ and $x \in V$.

5.2 Remark. It is obvious that every α-regular space is ξ-regular but not conversely.

5.3 Lemma. The following properties are equivalent for a space X:

(a) X is ξ-regular.
(b) For each $x \in X$ and each open set U of X containing x, there exists $V \in \xi O(X)$ such that $x \in V \subset \xi Cl(V) \subset U$.
(c) For each closed set F of X, $\cap \{\xi Cl(V) / F \subset V \in \xi O(X)\} = F$.
(d) For each subset A of X and each open set U of X such that $A \cap U \neq \phi$, there exists $V \in \xi O(X)$ such that $A \cap V \neq \phi$ and $\xi Cl(V) \subset U$.
(e) For each non empty subset A of X and each closed subset F of X such that $A \cap F = \phi$, there exist $V, W \in \xi O(X)$ such that $A \cap V \neq \phi$, $F \subset W$ and $V \cap W \neq \phi$.

Proof.

(a) ⇒ (b). Let U be an open set containing x, then $X - U$ is closed in X and $x \notin X - U$. By (a), there exist $W, V \in \xi O(X)$ such that $x \in V$, $X - U \subset W$ and $V \cap W = \phi$.By **Lemma 2.2**, we have $\xi Cl(V) \cap W = \phi$ and hence $x \in V \subset \xi Cl(V) \subset U$.

(b) ⇒ (c). Let F be a closed set of X. If $F \subset V$, then by **Lemma 2.2 (iii)**, $\xi Cl(F) \subset \xi Cl(V)$ which gives $F \subset \xi Cl(V)$ as $F \subset \xi Cl(F)$. Therefore, $\cap \{\xi Cl(V) / F \subset V \in \xi O(X)\} \supset F$.

Conversely, let $x \notin F$. Then $X - F$ is an open set containing x. By (b), there exists $U \in \xi O(X)$ such that $x \in U \subset \xi Cl(U) \subset X - F$. Put $V = X - \xi Cl(U)$. By **Lemma 2.2,** $F \subset V \in \xi O(X)$ and $x \notin \xi Cl(V)$. This implies that $\cap \{\xi Cl(V) / F \subset V \in \xi O(X)\} \subset F$.

Hence $\cap \{\xi Cl(V) / F \subset V \in \xi O(X)\} = F$.

(c) ⇒ (d). Let A be a subset of X and let U be open in X such that $A \cap U \neq \phi$. Let $x \in A \cap U$, then $X - U$ is a closed set not containing x. By (c), there exists $W \in \xi O(X)$ such that $X - U \subset W$ and $x \notin \xi Cl(W)$. Put $V = X - \xi Cl(W)$. Then $V \subset X - W$. Also $x \in V \cap A$. By using **Lemma 2.2**, we obtain $V \in \xi O(X)$, and $\xi Cl(V) \subset \xi Cl(X - W) = X - W \subset U$.

(d) ⇒ (e). Let A be a subset of X and let F be a closed set in X such that $A \cap F = \phi$, where $A \neq \phi$. Since $X - F$ is open in X and $A \neq \phi$, by (d), there exists $V \in \xi O(X)$ such that $A \cap V \neq \phi$ and $\xi C1(V) \subset X - F$. Put $W = X - \xi C1(V)$, then $F \subset W$. Also, $V \cap W = \phi$. By **Lemma 2.2**, $W \in \xi O(X)$.

(e) ⇒ (a). This is obvious.

5.4 Theorem. The following properties are equivalent for a space X:

(a) X is ξ-regular.
(b) For each closed set F and each point $x \in X - F$, there exists $U \in \xi O(X)$ and a gξ-open set V such that $x \in U$ and $F \subset V$ and $U \cap V = \phi$.
(c) For each subset A of X and each closed set F such that $A \cap F = \phi$, there exist $U \in \xi O(X)$ and a gξ-open set V such that $A \cap U \neq \phi$, $F \subset V$ and $U \cap V = \phi$.
(d) For each closed set F of X, $F = \cap \{\xi C1(V): F \subset V$ and V is gξ-open}.

Proof.

(a) ⇒ (b). The proof is obvious since every ξ-open set is gξ-open.

(b) ⇒ (c). Let A be a subset of X and let F be a closed set in X such that $A \cap F = \phi$. For a point $x \in A$, $x \in X - F$ and hence there exists $U \in \xi O(X)$ and a gξ-open set V such that $x \in U$ and $F \subset V$ and $U \cap V = \phi$. Also $x \in A$, $x \in U$ implies $x \in A \cap U$. So $A \cap U \neq \phi$.

(c) ⇒ (a). Let F be a closed set and let $x \in X - F$. Then, $\{x\} \cap F = \phi$ and there exist $U \in \xi O(X)$ and a gξ-open set W such that $x \in U$, $F \subset W$ and $U \cap W = \phi$. Put $V = \xi Int(W)$, then by **Lemma 2.9**, we have $F \subset V$, $V \in \xi O(X)$ and $U \cap V = \phi$. Therefore X is ξ-regular.

(a) ⇒ (d). For a closed set F of X, by **Lemma 5.3**, we obtain

$F \subset \cap \{\xi C1(V): F \subset V$ and V is gξ-open}
$\subset \cap \{\xi C1(V): F \subset V$ and $V \in \xi O(X)\} = F$

Therefore, $F = \cap \{\xi C1(V): F \subset V$ and V is gξ-open}.

(d) ⇒ (a). Let F be a closed set of X and $x \in X - F$. by (d), there exists a gξ-open set W of X such that $F \subset W$ and $x \in X - \xi C1(W)$. Since F is closed, $F \subset \xi Int(W)$ by **Lemma 2.9**. Put $V = \xi Int(W)$, then $F \subset V$ and $V \in \xi O(X)$. Since $x \in X - \xi C1(W)$, $x \in X - \xi C1(V)$. Put $U = X - \xi C1(V)$ then, $x \in U$, $U \in \xi O(X)$ and $U \cap V = \phi$. This shows that X is ξ-regular.

5.5 Definition. A function f: X → Y is said to be **ξ-open** [2] if for each open set U of X, $f(U) \in \xi O(Y)$.

5.6 Theorem. If f: X → Y is a continuous ξ-open gξ-closed surjection and X is regular, then Y is ξ-regular.

Proof. Let $y \in Y$ and let V be an open set of Y containing y. Let x be a point of X such that $y = f(x)$. By the regularity of X, there exists an open set U of X such that $x \in U \subset C1(U) \subset f^{-1}(V)$. We have $y \in f(U) \subset f(C1(U)) \subset V$. since f is ξ-open and gξ-closed, $f(U) \in \xi O(Y)$ and f(C1(U)) is gξ-closed in Y. So, we obtain, $y \in f(U) \subset \xi C1(f(U)) \subset \xi Cl(f(C1(U))) \subset V$. It follows from **Lemma 5.4** that Y is ξ-regular.

5.7 Definition. A function f: X → Y is said to be **pre ξ-open** if for each ξ-open set U of X, $f(U) \in \xi O(Y)$.

5.8 Theorem. If f: X → Y is a continuous pre ξ-open ξ-gξ-closed surjection and X is ξ-regular, then Y is ξ-regular.

Proof. Let F be any closed set of Y and $y \in Y - F$. Then $f^{-1}(Y) \cap f^{-1}(F) = \phi$ and $f^{-1}(F)$ is closed in X. Since X is ξ-regular, for a point $x \in f^{-1}(y)$, there exist $U, V \in \xi O(X)$ such that $x \in U$, $f^{-1}(F) \subset V$ and $U \cap V = \phi$. Since F is closed in Y, by **Proposition 3.6**, there exists $W \in \xi O(Y)$ such that $F \subset W$ and $f^{-1}(W) \subset V$. Since f pre ξ-open, we have $y = f(x) \in f(U)$ and $f(U) \in \xi O(Y)$. Since $U \cap V = \phi$, $f^{-1}(W) \cap U = \phi$ and hence $W \cap f(U) = \phi$. This shows that Y is ξ-regular.

6. Conclusion

We introduced a weaker version of normality called ξ-normality in topological spaces. We gave some characterizations and preservation theorems of ξ-normal and ξ-regular spaces. Some counterexamples were given and some basic properties were presented. The relationships among normal, α-normal, p-normal, β-normal, and ξ-normal are investigated.

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