

ξ -Normal and ξ -Regular Spaces in Topological Spaces

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Abstract: The aim of this paper is to introduce and study two new classes of spaces, namely ξ -normal and ξ -regular spaces in topological spaces. The relationships among normal, p -normal, α -normal, β -normal and ξ -normal spaces are investigated. Moreover, we introduced some functions such as $g\xi$ -closed, ξ - $g\xi$ -closed, pre ξ -open. We obtained some characterizations of ξ -normal and ξ -regular spaces, properties of the forms of $g\xi$ -closed functions and preservation theorems for ξ -normal and ξ -regular spaces.

Keywords: ξ -closed sets, ξ -normal, ξ -regular spaces, $g\xi$ -closed and ξ - $g\xi$ -closed functions

2010 Mathematics Subject Classification: 54D10, 54D15, 54A05, 54C08.

1. Introduction

Levine [3] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. α -open sets were introduced by Njastad [7]. Devi et al. [2] introduced the concept of ξ -closed sets. Nour [8] introduced the notion of p -normal spaces and obtained their characterizations and preservation theorems. Paul and Bhattacharyya [9] obtained some properties of p -normal spaces. Benchalli et al. [1] introduced the notion of α -normal spaces and obtained their characterizations and preservation theorems. Mahmoud et al. [4] introduced the notion of β -normal spaces and obtained their characterizations and preservation theorems. Recently, Sharma et al. [10] introduced a new class of regular spaces called ξ -regular spaces by using ξ -open sets introduced by Devi et al. [2] and obtained several properties such as characterizations and preservation theorems for ξ -regular spaces.

2. Preliminaries

Throughout this paper, spaces (X, τ) , (Y, σ) , and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X . The closure of A and interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. A is said to be α -open [1] if $A \subset Int(Cl(Int(A)))$. The complement of a α -open set is said to be α -closed [1]. The intersection of all α -closed sets containing A is called α -closure [2] of A , and is denoted by $\alpha Cl(A)$.

2.1 Definition. A subset A of a space (X, τ) is said to be

1. generalized closed (briefly **g -closed**) [3] if $Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

2. generalized α -closed (briefly **αg -closed**) [6] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

3. generalized α -closed (briefly **$g\alpha$ -closed**) [5] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is α -open in X .

4. ξ -closed [2] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is $g\alpha$ -open in X .

5. g -open (resp. **αg -open**, **$g\alpha$ -open**, **ξ -open**) if the complement of A is g -closed (resp. αg -closed, $g\alpha$ -closed, ξ -closed).

The intersection of all ξ -closed sets containing A is called **ξ -closure** of A , and is denoted by $\xi Cl(A)$. The **ξ -interior** of A , denoted by $\xi Int(A)$, is defined as union of all ξ -open sets contained in A . The family of all ξ -closed (resp. ξ -open) sets of a space X is denoted by $\xi C(X)$ (resp. $\xi O(X)$).

2.2 Lemma. Let A be a subset of a space X and $x \in X$. The following properties hold for $\xi Cl(A)$:

- $x \in \xi Cl(A)$ if and only if $A \cap U \neq \emptyset$ for every $U \in \xi O(X)$ containing x .
- A is ξ -closed if and only if $A = \xi Cl(A)$.
- $\xi Cl(A) \subset \xi Cl(B)$ if $A \subset B$.
- $\xi Cl(\xi Cl(A)) = \xi Cl(A)$.
- $\xi Cl(A)$ is ξ -closed.

2.3 Definition. A subset A of a space X is said to be **generalized ξ -closed** (briefly **$g\xi$ -closed**) if $\xi Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

2.4 Remark. We have the following implications for the properties of subsets:

$$\begin{aligned} \text{closed} &\Rightarrow g\text{-closed} \\ \Downarrow \Downarrow \\ \alpha\text{-closed} &\Rightarrow \alpha g\text{-closed} \\ \Downarrow \Downarrow \\ \xi\text{-closed} &\Rightarrow g\xi\text{-closed} \end{aligned}$$

Where none of the implications is reversible as can be seen from the following examples:

2.5 Example Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $A = \{b\}$ is g -closed but not closed.

2.6 Example. Let $X = \{a, b, c, \}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $A = \{a, b\}$ is $g\xi$ -closed as well as $g\xi$ -closed.

2.7 Example Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$ Then $A = \{a\}$ is α -closed as well as ξ -closed but not closed.

2.8 Example Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c, d\}, X\}$. Then $A = \{a, b, c\}$ is ξ -closed. But it is neither α -closed nor closed.

2.9 Lemma. A subset A of a space X is $g\xi$ -open in X if and only if $F \subset \xi\text{Int}(A)$ whenever $F \subset A$ and F is closed in X .

3. Generalized ξ -closed functions

3.1 Definition. A function $f: X \rightarrow Y$ is said to be **ξ -closed** [2] if for each closed set F of X , $f(F)$ is ξ -closed in Y .

3.2 Definition. A function $f: X \rightarrow Y$ is said to be

- (i) **generalized ξ -closed** (briefly **$g\xi$ -closed**) if for each closed set F of X , $f(F)$ is $g\xi$ -closed in Y .
- (ii) **ξ -generalized ξ -closed** (briefly **ξ - $g\xi$ -closed**) if for each ξ -closed set F of X , $f(F)$ is $g\xi$ -closed in Y .

3.3 Remark. Every closed function is ξ -closed but not conversely. Also, every ξ -closed function is $g\xi$ -closed because every ξ -closed set is $g\xi$ -closed. It is obvious that both ξ -closedness and ξ - $g\xi$ -closedness imply $g\xi$ -closedness.

3.4 Theorem. A surjective function $f: X \rightarrow Y$ is $g\xi$ -closed (resp. ξ - $g\xi$ -closed) if and only if for each subset B of Y and each open (resp. ξ -open) set U of X containing $f^{-1}(B)$, there exists a $g\xi$ -open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose that f is $g\xi$ -closed (resp. ξ - $g\xi$ -closed). Let B be any subset of Y and U be open (resp. ξ -open) set of X containing $f^{-1}(B)$. Put $V = Y - f(X - U)$. Then the complement V^c of V is $V^c = Y - V = f(X - U)$. Since $X - U$ is closed in X and f is $g\xi$ -closed, $f(X - U) = V^c$ is $g\xi$ -closed. Therefore, V is $g\xi$ -open in Y . It is easy to see that $B \subset V$ and $f^{-1}(V) \subset U$.

Conversely, let F be a closed (resp. ξ -closed) set of X . Put $B = Y - f(F)$, then we have $f^{-1}(B) \subset X - F$ and $X - F$ is open (resp. ξ -open) in X . Then by assumption, there exists a $g\xi$ -open set V of Y such that $B = Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Now $f^{-1}(V) \subset X - F$ implies $V \subset Y - f(F) = B$. Also $B \subset V$ and so $B = V$. Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is $g\xi$ -closed in Y . This shows that f is $g\xi$ -closed (resp. ξ - $g\xi$ -closed).

3.5 Remark. We can prove the necessity part of the above theorem by replacing each set to closed set in the form of the proposition given below:

3.6 Proposition. If a surjective function $f: X \rightarrow Y$ is $g\xi$ -closed (resp. ξ - $g\xi$ -closed) then for a closed set F of Y and for any open (resp. ξ -open) set U of X containing $f^{-1}(F)$, there exists a ξ -open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$.

Proof. By **Theorem 3.4**, there exists a $g\xi$ -open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset U$. Since F is closed, by **Lemma 2.9** we have $F \subset \xi\text{Int}(W)$. Put $V = \xi\text{Int}(W)$. Then $V \in \xi\text{O}(Y)$, $F \subset V$ and $f^{-1}(V) \subset U$.

3.7 Proposition. If $f: X \rightarrow Y$ is continuous ξ - $g\xi$ -closed and A is $g\xi$ -closed in X , then $f(A)$ is $g\xi$ -closed in Y .

Proof. Let V be a open set of Y containing $f(A)$. Then $A \subset f^{-1}(V)$. Since f is continuous, $f^{-1}(V)$ is open in X . Since A is $g\xi$ -closed in X , by a definition, we get $\xi\text{C1}(A) \subset f^{-1}(V)$ and hence $f(\xi\text{C1}(A)) \subset V$. Since f is ξ - $g\xi$ -closed and $\xi\text{C1}(A)$ is ξ -closed in X , $f(\xi\text{C1}(A))$ is $g\xi$ -closed in Y and hence we have $\xi\text{C1}(f(\xi\text{C1}(A))) \subset V$. By definition of the ξ -closure of a set, $A \subset \xi\text{C1}(A)$ which implies $f(A) \subset f(\xi\text{C1}(A))$ and using **Lemma 2.2**, $\xi\text{C1}(f(A)) \subset \xi\text{C1}(f(\xi\text{C1}(A))) \subset V$. That is $\xi\text{C1}(f(A)) \subset V$. This shows that $f(A)$ is $g\xi$ -closed in Y .

3.8 Definition. A function $f: X \rightarrow Y$ is said to be **ξ -irresolute** [2] if for each $V \in \xi\text{O}(Y)$, $f^{-1}(V) \in \xi\text{O}(X)$.

3.9 Proposition. If $f: X \rightarrow Y$ is an open ξ -irresolute bijection and B is $g\xi$ -closed in Y , then $f^{-1}(B)$ is $g\xi$ -closed in X .

Proof. Let U be a open set of X containing $f^{-1}(B)$. Then $B \subset f(U)$ and $f(U)$ is open in Y . Since B is $g\xi$ -closed in Y , $\xi\text{C1}(B) \subset f(U)$ and hence we have $f^{-1}(\xi\text{C1}(B)) \subset U$. Since f is ξ -irresolute, $f^{-1}(\xi\text{C1}(B))$ is ξ -closed in X (**Theorem 2.1** (i) and (v)), we have $\xi\text{C1}(f^{-1}(B)) \subset f^{-1}(\xi\text{C1}(B)) \subset U$. This shows that $f^{-1}(B)$ is $g\xi$ -closed in X .

3.10 Theorem. Let $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ be the two functions, then

- (i) If $h \circ f: X \rightarrow Z$ is $g\xi$ -closed and if $f: X \rightarrow Y$ is a continuous surjection, then $h: Y \rightarrow Z$ is $g\xi$ -closed.
- (ii) If $f: X \rightarrow Y$ is $g\xi$ -closed with $h: Y \rightarrow Z$ is continuous and ξ - $g\xi$ -closed, then $h \circ f: X \rightarrow Z$ is $g\xi$ -closed.
- (iii) If $f: X \rightarrow Y$ is closed and $h: Y \rightarrow Z$ is $g\xi$ -closed, then $h \circ f: X \rightarrow Z$ is $g\xi$ -closed.

Proof.

- (i) Let F be a closed set of Z . Then $f^{-1}(F)$ is closed in Y since f is continuous. By hypothesis ($h \circ f$) ($f^{-1}(F)$) is $g\xi$ -closed in Z . Hence h is $g\xi$ -closed.
- (ii) The proof follows from the **Proposition 3.7**.
- (iii) The proof is obvious from definitions.

4. ξ -Normal spaces

4.1 Definition. A space X is said to be ξ -normal (resp. α -normal [1], p -normal [8, 9], β -normal [4]) if for any pair of disjoint closed sets A, B of X , there exist disjoint ξ -open (resp. α -open, p -open, β -open) sets U and V such that $A \subset U$ and $B \subset V$.

By the definitions stated above, we have the following diagram:

normality \Rightarrow α -normality \Rightarrow p -normality \Rightarrow β -normality

\Downarrow

ξ -normality

Where none of the implications is reversible as can be seen from the following examples:

4.2 Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. The pair of disjoint closed subsets of X are $A = \{a\}$ and $B = \{c\}$. Taking β -open sets, $U = \{a, b\}$ and $V = \{c, d\}$ such that $A \subset U$ and $B \subset V$. Hence the space X is β -normal. But the space X is neither p -normal nor α -normal, since the sets U and V are neither p -open nor α -open..

4.3 Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. The pair of disjoint closed subsets of X are $A = \{a\}$ and $B = \{c\}$. Taking p -open sets, $U = \{a, b\}$ and $V = \{c, d\}$ such that $A \subset U$ and $B \subset V$. Hence the space X is p -normal as well as β -normal, since every p -open sets are β -open. But the space X is neither normal nor α -normal, since the sets U and V are neither open nor α -open.

4.4 Theorem. The following properties are equivalent for a space X :

- X is ξ -normal.
- For each pair of disjoint closed sets A, B of X , there exist disjoint $g\xi$ -open sets U and V such that $A \subset U$ and $B \subset V$.
- For each closed set A and any open set V containing A , there exists a $g\xi$ -open set U such that $A \subset U \subset \xi C1(U) \subset V$.
- For each closed set A and any g -open set B containing A , there exists a $g\xi$ -open set U such that $A \subset U \subset \xi C1(U) \subset \text{Int}(B)$.
- For each closed set A and any g -open set B containing A , there exists a ξ -open set G such that $A \subset G \subset \xi C1(G) \subset \text{Int}(B)$.
- For each g -closed set A and any open set B containing A , there exists a ξ -open set U such that $C1(A) \subset U \subset \xi C1(U) \subset B$.
- For each g -closed set A and any open set B containing A , there exists a $g\xi$ -open set G such that $C1(A) \subset G \subset \xi C1(G) \subset B$.

Proof. (a) \Rightarrow (b). This proof is obvious since every ξ -open set is $g\xi$ -open.

(b) \Rightarrow (c). Let A be a closed set and let V be an open set containing A . Since A and $X - V$ are disjoint closed sets of X , there exist $g\xi$ -open sets U and W of X such that $A \subset U$ and $X - V \subset W$ and $U \cap W = \emptyset$. By **Lemma 2.9**, we get $X - V \subset \xi \text{Int}(W)$. Since $U \cap \xi \text{Int}(W) = \emptyset$, we have $\xi C1(U) \cap \xi \text{Int}(W) = \emptyset$ and hence $\xi C1(U) \subset X - \xi \text{Int}(W) \subset V$. Therefore, we obtain $A \subset U \subset \xi C1(U) \subset V$.

(c) \Rightarrow (a). Let A and B be the disjoint closed sets of X . Since $X - B$ is an open set containing A , there exists a $g\xi$ -open set G such that $A \subset G \subset \xi C1(G) \subset X - B$. By **Lemma 2.9**, we have $A \subset \xi \text{Int}(G)$. Put $U = \xi \text{Int}(G)$ and $V = X - \xi C1(G)$. Then U and V are disjoint ξ -open sets such that $A \subset U$ and $B \subset V$. Therefore X is ξ -normal.

Since every ξ -open set is $g\xi$ -open and every closed (resp. open) set is g -closed (resp. g -open), it is obvious that (d) \Rightarrow (e) \Rightarrow (c) and (f) \Rightarrow (g) \Rightarrow (c).

(c) \Rightarrow (e). Let A be a closed set of X and let B be a g -open set such that $A \subset B$. Since B is g -open and A is closed, $A \subset \text{Int}(B)$ by **Lemma 2.9**. Therefore by (c), there exists a $g\xi$ -open set U such that $A \subset U \subset \xi C1(U) \subset \xi \text{Int}(B)$.

(e) \Rightarrow (d). Let A be a closed set of X and let B be a g -open set such that $A \subset B$. Then there exists a $g\xi$ -open set G such that $A \subset G \subset \xi C1(G) \subset \text{Int}(B)$ by **Lemma 2.9**. Since G is $g\xi$ -open, $A \subset \xi \text{Int}(G)$. Put $U = \xi \text{Int}(G)$, then U is ξ -open and $A \subset U \subset \xi C1(U) \subset \text{Int}(B)$.

(c) \Rightarrow (g). Let A be a g -closed set of X and let B be an open set such that $A \subset B$. Then $C1(A) \subset B$. Therefore by (c), there exists a $g\xi$ -open set U such that $C1(A) \subset U \subset \xi C1(U) \subset B$.

(g) \Rightarrow (f). Let A be a g -closed set of X and let B be an open set such that $A \subset B$. Then there exist a $g\xi$ -open set G such that $C1(A) \subset G \subset \xi C1(G) \subset B$. Since G is $g\xi$ -open and the closed set $C1(A) \subset G$, we have $C1(A) \subset \xi \text{Int}(G)$ by **Lemma 2.9**. Put $U = \xi \text{Int}(G)$. Then, U is ξ -open and $C1(A) \subset U \subset \xi C1(U) \subset B$.

4.5 Theorem. If $f: X \rightarrow Y$ is continuous $g\xi$ -closed surjection and X is normal, then Y is ξ -normal.

Proof. Let A and B be the disjoint closed sets of Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X since f is continuous. Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By **Proposition 3.6**, there exist $g\xi$ -open sets G and H of Y such that $A \subset G$, $B \subset H$ and $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and hence $G \cap H = \emptyset$. It follows from **Theorem 4.4** that Y is ξ -normal.

4.6 Theorem. If $f: X \rightarrow Y$ is continuous ξ - $g\xi$ -closed surjection and X is ξ -normal, then Y is ξ -normal.

Proof. Let A and B be disjoint closed sets of Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X . Since X is ξ -normal, there exist disjoint ξ -open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since f is ξ - $g\xi$ -closed, by **Proposition 3.6**, there exist ξ -open sets G and H of Y such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Since U and V are disjoint, we have $G \cap H = \phi$. This shows that Y is ξ -normal.

5. ξ -Regular Spaces

5.1 Definition. A space X is said to be ξ -regular [10] (resp. α -regular [1]) if for each closed set F of X , and each point $x \in X - F$, there exist disjoint ξ -open (resp. α -open) set U, V such that $F \subset U$ and $x \in V$.

5.2 Remark. It is obvious that every α -regular space is ξ -regular but not conversely.

5.3 Lemma. The following properties are equivalent for a space X :

- (a) X is ξ -regular.
- (b) For each $x \in X$ and each open set U of X containing x , there exists $V \in \xi O(X)$ such that $x \in V \subset \xi Cl(V) \subset U$.
- (c) For each closed set F of X , $\bigcap \{ \xi Cl(V) / F \subset V \in \xi O(X) \} = F$.
- (d) For each subset A of X and each open set U of X such that $A \cap U \neq \phi$, there exists $V \in \xi O(X)$ such that $A \cap V \neq \phi$ and $\xi Cl(V) \subset U$.
- (e) For each non empty subset A of X and each closed subset F of X such that $A \cap F = \phi$, there exist $V, W \in \xi O(X)$ such that $A \cap V \neq \phi, F \subset W$ and $V \cap W = \phi$.

Proof.

(a) \Rightarrow (b). Let U be an open set containing x , then $X - U$ is closed in X and $x \notin X - U$. By (a), there exist $W, V \in \xi O(X)$ such that $x \in V, X - U \subset W$ and $V \cap W = \phi$. By **Lemma 2.2**, we have $\xi Cl(V) \cap W = \phi$ and hence $x \in V \subset \xi Cl(V) \subset U$.

(b) \Rightarrow (c). Let F be a closed set of X . If $F \subset V$, then by **Lemma 2.2 (iii)**, $\xi Cl(F) \subset \xi Cl(V)$ which gives $F \subset \xi Cl(V)$ as $F \subset \xi Cl(F)$. Therefore, $\bigcap \{ \xi Cl(V) / F \subset V \in \xi O(X) \} \supset F$.

Conversely, let $x \notin F$. Then $X - F$ is an open set containing x . By (b), there exists $U \in \xi O(X)$ such that $x \in U \subset \xi Cl(U) \subset X - F$. Put $V = X - \xi Cl(U)$. By **Lemma 2.2**, $F \subset V \in \xi O(X)$ and $x \notin \xi Cl(V)$. This implies that $\bigcap \{ \xi Cl(V) / F \subset V \in \xi O(X) \} \subset F$.

Hence $\bigcap \{ \xi Cl(V) / F \subset V \in \xi O(X) \} = F$.

(c) \Rightarrow (d). Let A be a subset of X and let U be open in X such that $A \cap U \neq \phi$. Let $x \in A \cap U$, then $X - U$ is a closed set not containing x . By (c), there exists $W \in \xi O(X)$ such that $X - U \subset W$ and $x \notin \xi Cl(W)$. Put $V = X - \xi Cl(W)$. Then $V \subset X - W$. Also $x \in V \cap A$. By using

Lemma 2.2, we obtain $V \in \xi O(X)$, and $\xi Cl(V) \subset \xi Cl(X - W) = X - W \subset U$.

(d) \Rightarrow (e). Let A be a subset of X and let F be a closed set in X such that $A \cap F = \phi$, where $A \neq \phi$. Since $X - F$ is open in X and $A \neq \phi$, by (d), there exists $V \in \xi O(X)$ such that $A \cap V \neq \phi$ and $\xi Cl(V) \subset X - F$. Put $W = X - \xi Cl(V)$, then $F \subset W$. Also, $V \cap W = \phi$. By **Lemma 2.2**, $W \in \xi O(X)$.

(e) \Rightarrow (a). This is obvious.

5.4 Theorem. The following properties are equivalent for a space X :

- (a) X is ξ -regular.
- (b) For each closed set F and each point $x \in X - F$, there exists $U \in \xi O(X)$ and a $g\xi$ -open set V such that $x \in U$ and $F \subset V$ and $U \cap V = \phi$.
- (c) For each subset A of X and each closed set F such that $A \cap F = \phi$, there exist $U \in \xi O(X)$ and a $g\xi$ -open set V such that $A \cap U \neq \phi, F \subset V$ and $U \cap V = \phi$.
- (d) For each closed set F of X , $F = \bigcap \{ \xi Cl(V) : F \subset V \text{ and } V \text{ is } g\xi\text{-open} \}$.

Proof.

(a) \Rightarrow (b). The proof is obvious since every ξ -open set is $g\xi$ -open.

(b) \Rightarrow (c). Let A be a subset of X and let F be a closed set in X such that $A \cap F = \phi$. For a point $x \in A, x \in X - F$ and hence there exists $U \in \xi O(X)$ and a $g\xi$ -open set V such that $x \in U$ and $F \subset V$ and $U \cap V = \phi$. Also $x \in A, x \in U$ implies $x \in A \cap U$. So $A \cap U \neq \phi$.

(c) \Rightarrow (a). Let F be a closed set and let $x \in X - F$. Then, $\{x\} \cap F = \phi$ and there exist $U \in \xi O(X)$ and a $g\xi$ -open set W such that $x \in U, F \subset W$ and $U \cap W = \phi$. Put $V = \xi Int(W)$, then by **Lemma 2.9**, we have $F \subset V, V \in \xi O(X)$ and $U \cap V = \phi$. Therefore X is ξ -regular.

(a) \Rightarrow (d). For a closed set F of X , by **Lemma 5.3**, we obtain

$$F \subset \bigcap \{ \xi Cl(V) : F \subset V \text{ and } V \text{ is } g\xi\text{-open} \} \\ \subset \bigcap \{ \xi Cl(V) : F \subset V \text{ and } V \in \xi O(X) \} = F$$

Therefore, $F = \bigcap \{ \xi Cl(V) : F \subset V \text{ and } V \text{ is } g\xi\text{-open} \}$.

(d) \Rightarrow (a). Let F be a closed set of X and $x \in X - F$. By (d), there exists a $g\xi$ -open set W of X such that $F \subset W$ and $x \in X - \xi Cl(W)$. Since F is closed, $F \subset \xi Int(W)$ by **Lemma 2.9**. Put $V = \xi Int(W)$, then $F \subset V$ and $V \in \xi O(X)$. Since $x \in X - \xi Cl(W), x \in X - \xi Cl(V)$. Put $U = X - \xi Cl(V)$ then, $x \in U, U \in \xi O(X)$ and $U \cap V = \phi$. This shows that X is ξ -regular.

5.5 Definition. A function $f: X \rightarrow Y$ is said to be ξ -open [2] if for each open set U of $X, f(U) \in \xi O(Y)$.

5.6 Theorem. If $f: X \rightarrow Y$ is a continuous ξ -open $g\xi$ -closed surjection and X is regular, then Y is ξ -regular.

Proof. Let $y \in Y$ and let V be an open set of Y containing y . Let x be a point of X such that $y = f(x)$. By the regularity of X , there exists an open set U of X such that $x \in U \subset C1(U) \subset f^{-1}(V)$. We have $y \in f(U) \subset f(C1(U)) \subset V$. since f is ξ -open and $g\xi$ -closed, $f(U) \in \xi O(Y)$ and $f(C1(U))$ is $g\xi$ -closed in Y . So, we obtain, $y \in f(U) \subset \xi C1(f(U)) \subset \xi Cl(f(C1(U))) \subset V$. It follows from **Lemma 5.4** that Y is ξ -regular.

5.7 Definition. A function $f: X \rightarrow Y$ is said to be **pre ξ -open** if for each ξ -open set U of X , $f(U) \in \xi O(Y)$.

5.8 Theorem. If $f: X \rightarrow Y$ is a continuous pre ξ -open $g\xi$ -closed surjection and X is ξ -regular, then Y is ξ -regular.

Proof. Let F be any closed set of Y and $y \in Y - F$. Then $f^{-1}(Y) \cap f^{-1}(F) = \phi$ and $f^{-1}(F)$ is closed in X . Since X is ξ -regular, for a point $x \in f^{-1}(y)$, there exist $U, V \in \xi O(X)$ such that $x \in U, f^{-1}(F) \subset V$ and $U \cap V = \phi$. Since F is closed in Y , by **Proposition 3.6**, there exists $W \in \xi O(Y)$ such that $F \subset W$ and $f^{-1}(W) \subset V$. Since f pre ξ -open, we have $y = f(x) \in f(U)$ and $f(U) \in \xi O(Y)$. Since $U \cap V = \phi, f^{-1}(W) \cap U = \phi$ and hence $W \cap f(U) = \phi$. This shows that Y is ξ -regular.

6. Conclusion

We introduced a weaker version of normality called ξ -normality in topological spaces. We gave some characterizations and preservation theorems of ξ -normal and ξ -regular spaces. Some counterexamples were given and some basic properties were presented. The relationships among normal, α -normal, p -normal, β -normal, and ξ -normal are investigated.

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