ξ-Normal and ξ-Regular Spaces in Topological Spaces

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Abstract: The aim of this paper is to introduce and study two new classes of spaces, namely ξ-normal and ξ-regular spaces in topological spaces. The relationships among normal, p-normal, ξ-normal, β-normal and ξ-normal spaces are investigated. Moreover, we introduced some functions such as gξ-closed, ξgξ-closed, pre ξ-open. We obtained several characterizations of ξ-normal and ξ-regular spaces, properties of the forms of gξ-closed functions and preservation theorems for ξ-normal and ξ-regular spaces.

Keywords: ξ-closed sets, ξ-normal, ξ-regular spaces, gξ-closed and ξ-gξ-closed functions

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1. Introduction

Levine [3] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. α-open sets were introduced by Njastad [7]. Devi et al. [2] introduced the concept of ξ-closed sets. Nour [8] introduced the notion of p-normal spaces and obtained their characterizations and preservation theorems. Paul and Bhattacharyya [9] obtained some properties of p-normal spaces. Benchali et al. [1] introduced the notion of α-normal spaces and obtained their characterizations and preservation theorems. Mahmoud et al. [4] introduced the notion of β-normal spaces and obtained their characterizations and preservation theorems. Recently, Sharma et al. [10] introduced a new class of regular spaces called ξ-regular spaces by using ξ-open sets introduced by Devi et al. [2] and obtained several properties such as characterizations and preservation theorems for ξ-regular spaces.

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by Cl(A) and Int(A) respectively. A is said to be α-open [1] if A ⊂ Int(Cl(A)). The complement of a α-open set is said to be α-closed [2]. The intersection of all α-closed sets containing A is called α-closure [2] of A, and is denoted by αCl(A).

2.1 Definition. A subset A of a space (X, τ) is said to be

1. generalized closed (briefly g-closed) [3] if Cl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
2. generalized α-closed (briefly αg-closed) [6] if αCl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
3. generalized α-closed (briefly αg-closed) [5] if αCl(A) ⊂ U whenever A ⊂ U and U is α-open in X.

4. ξ-closed [2] if αCl(A) ⊂ U whenever A ⊂ U and U is gα-open in X.
5. g-open (resp. αg-open, gα-open, ξ-open) if the complement of A is g-closed (resp. αg-closed, gα-closed, ξ-closed).

The intersection of all ξ-closed sets containing A is called ξ-closure of A, and is denoted by ξCl(A). The ξ-interior of A, denoted by ξInt(A), is defined as union of all ξ-open sets contained in A. The family of all ξ-closed (resp. ξ-open) sets of a space X is denoted by ξC(X) (resp. ξO(X)).

2.2 Lemma. Let A be a subset of a space X and x ∈ X. The following properties hold for ξCl(A):

(i) x ∈ ξCl(A) if and only if A ∩ U ≠ ∅ for every U ∈ ξO(X) containing x.
(ii) A is ξ-closed if and only if A = ξCl(A).
(iii) ξCl(A) ⊂ ξCl(B) if A ⊂ B.
(iv) ξCl(ξCl(A)) = ξCl(A).
(v) ξCl(A) is ξ-closed.

2.3 Definition. A subset A of a space X is said to be generalized ξ-closed (briefly gξ-closed) if ξCl(A) ⊂ U whenever A ⊂ U and U ∈ τ.

2.4 Remark. We have the following implications for the properties of subsets:

closed ⇒ g-closed
⇒ α-closed ⇒ αg-closed
⇒ ξ-closed ⇒ gξ-closed

Where none of the implications is reversible as can be seen from the following examples:

2.5 Example Let X = {a, b, c} and τ = {∅, {a}, X}. Then A = {b} is g-closed but not closed.
2.6 Example. Let $X = \{a, b, c, \}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $A = \{a, b\}$ is g-closed as well as g-$\xi$-closed.

2.7 Example Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$ Then $A = \{a\}$ is $\alpha$-closed as well as $\xi$-closed but not closed.

2.8 Example Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c, d\}, X\}$. Then $A = \{a, b, c\}$ is $\xi$-closed. But it is neither $\alpha$-closed nor closed.

2.9 Lemma. A subset $A$ of a space $X$ is g-$\xi$-open in $X$ if and only if $F \subset \xi \text{Int}(A)$ whenever $F \subset \text{A}$ and $F$ is closed in $X$.

3. Generalized $\xi$-closed functions

3.1 Definition. A function $f: X \rightarrow Y$ is said to be $\xi$-closed [2] if for each closed set $F$ of $X$, $f(F)$ is $\xi$-closed in $Y$.

3.2 Definition. A function $f: X \rightarrow Y$ is said to be

(i) generalized $\xi$-closed (briefly g-$\xi$-closed) if for each closed set $F$ of $X$, $f(F)$ is g-$\xi$-closed in $Y$.

(ii) $\xi$-generalized $\xi$-closed (briefly $\xi$-g-$\xi$-closed) if for each $\xi$-closed set $F$ of $X$, $f(F)$ is g-$\xi$-closed in $Y$.

3.3 Remark. Every closed function is $\xi$-closed but not conversely. Also, every $\xi$-closed function is g-$\xi$-closed because every $\xi$-closed set is g-$\xi$-closed. It is obvious that both $\xi$-closedness and $\xi$-g-$\xi$-closedness imply g-$\xi$-closedness.

3.4 Theorem. A surjective function $f: X \rightarrow Y$ is g-$\xi$-closed (resp. $\xi$-g-$\xi$-closed ) if and only if for each subset $B$ of $Y$ and each open (resp. $\xi$-open ) set $U$ of $X$ containing $f^{-1}(B)$, there exists a g-$\xi$-open set $V$ of $Y$ such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose that $f$ is g-$\xi$-closed ( resp. $\xi$-g-$\xi$-closed). Let $B$ be any subset of $Y$ and $U$ be open (resp. $\xi$-open) set of $X$ containing $f^{-1}(B)$. Put $V = Y - f(X - U)$. Then the complement $V^c$ of $V$ is $V^c = Y - V = f(X - U)$. Since $X - U$ is closed in $X$ and $f$ is g-$\xi$-closed, $f(X - U) = V^c$ is g-$\xi$-closed. Therefore, $V$ is g-$\xi$-open in $Y$. It is easy to see that $B \subset V$ and $f^{-1}(V) \subset U$.

Conversely, let $F$ be a closed (resp. $\xi$-closed) set of $X$. Put $B = Y - f(F)$, then we have $f^{-1}(B) \subset X - F$ and $X - F$ is open (resp. $\xi$-open) in $X$. Then by assumption, there exists a g-$\xi$-open set $V$ of $Y$ such that $B = Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Now $f^{-1}(V) \subset X - F$ implies $V \subset Y - f(F)$ and $f^{-1}(V) \subset X - F$. So there exists $B \subset Y$ such that $f^{-1}(V) \subset X - F$. Also $B \subset V$ and $B \subset f^{-1}(V)$. Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is g-$\xi$-closed in $Y$. This shows that $f$ is g-$\xi$-closed (resp. $\xi$-g-$\xi$-closed).

3.5 Remark. We can prove the necessity part of the above theorem by replacing each set to closed set in the form of the proposition given below:

3.6 Proposition. If a surjective function $f: X \rightarrow Y$ is g-$\xi$-closed (resp. $\xi$-g-$\xi$-closed) then for a closed set $F$ of $Y$ and for any open (resp. $\xi$-open) set $U$ of $X$ containing $f^{-1}(F)$, there exists a $\xi$-open set $V$ of $Y$ such that $F \subset V$ and $f^{-1}(V) \subset U$.

Proof. By Theorem 3.4, there exists a g-$\xi$-open set $W$ of $Y$ such that $F \subset W$ and $f^{-1}(W) \subset U$. Since $F$ is closed, by Lemma 2.9 we have $F \subset \xi \text{Int}(W)$. Put $V = \xi \text{Int}(W)$. Then $V \in \xi \text{Int}(Y)$, $F \subset V$ and $f^{-1}(V) \subset U$.

3.7 Proposition. If $f: X \rightarrow Y$ is continuous $\xi$-g-$\xi$-closed and $A$ is g-$\xi$-closed in $X$, then $f(A)$ is g-$\xi$-closed in $Y$.

Proof. Let $V$ be a open set of $Y$ containing $f(A)$. Then $A \subset f^{-1}(V)$. Since $f$ is continuous, $f^{-1}(V)$ is open in $X$. Since $A$ is g-$\xi$-closed in $X$, by a definition, we get $\xi \text{Cl}(A) \subset f^{-1}(V)$ and hence $f(\xi \text{Cl}(A)) \subset V$. Since $f$ is $\xi$-g-$\xi$-closed and $\xi \text{Cl}(A)$ is $\xi$-closed in $X$, $f(\xi \text{Cl}(A))$ is g-$\xi$-closed in $Y$ and hence we have $\xi \text{Cl}(f(\xi \text{Cl}(A))) \subset V$. By definition of the $\xi$-closure of a set, $A \subset \xi \text{Cl}(A)$ which implies $f(A) \subset f(\xi \text{Cl}(A))$ and using Lemma 2.2, $\xi \text{Cl}(f(A)) \subset \xi \text{Cl}(f(\xi \text{Cl}(A))) \subset V$. That is $\xi \text{Cl}(f(A)) \subset V$. This shows that $f(A)$ is g-$\xi$-closed in $Y$.

3.8 Definition. A function $f: X \rightarrow Y$ is said to be $\xi$-irresolute [2] if for each $V \in \xi \text{O}(Y)$, $f^{-1}(V) \in \xi \text{O}(X)$.

3.9 Proposition. If $f: X \rightarrow Y$ is an open $\xi$-irresolute bijection and $B$ is g-$\xi$-closed in $Y$, then $f^{-1}(B)$ is g-$\xi$-closed in $X$.

Proof. Let $U$ be an open set of $X$ containing $f^{-1}(B)$. Then $B \subset f(U)$ and $f(U)$ is open in $Y$. Since $B$ is g-$\xi$-closed in $Y$, $\xi \text{Cl}(B) \subset f(U)$ and hence we have $f^{-1}(\xi \text{Cl}(B)) \subset U$. Since $f$ is $\xi$-irresolute, $f^{-1}(\xi \text{Cl}(B))$ is $\xi$-closed in $X$ (Theorem 2.1 (i) and (iv)), we have $\xi \text{Cl}(f^{-1}(B)) \subset f^{-1}(\xi \text{Cl}(B)) \subset U$. This shows that $f^{-1}(B)$ is g-$\xi$-closed in $X$.

3.10 Theorem. Let $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ be the two functions, then

(i) If $h: X \rightarrow Z$ is g-$\xi$-closed and if $f: X \rightarrow Y$ is a continuous surjection, then $h: X \rightarrow Z$ is g-$\xi$-closed.

(ii) If $f: X \rightarrow Y$ is g-$\xi$-closed with $h: Y \rightarrow Z$ is continuous and $\xi$-g-$\xi$-closed, then $h: X \rightarrow Z$ is g-$\xi$-closed.

(iii) If $f: X \rightarrow Y$ is closed and $h: Y \rightarrow Z$ is g-$\xi$-closed, then $h: X \rightarrow Z$ is g-$\xi$-closed.

Proof.

(i) Let $F$ be a closed set of $Y$. Then $f^{-1}(F)$ is closed in $X$ since $f$ is continuous. By hypothesis (h), $(f^{-1}(F))$ is g-$\xi$-closed in $Z$. Hence h is g-$\xi$-closed.

(ii)The proof follows from the Proposition 3.7.

(iii)The proof is obvious from definitions.
4. $\xi$-Normal spaces

4.1 Definition. A space $X$ is said to be $\xi$-normal (resp. $\alpha$-normal [1], $p$-normal [8, 9], $\beta$-normal [4]) if for any pair of disjoint closed sets $A$, $B$ of $X$, there exist disjoint $\xi$-open (resp. $\alpha$-open, $p$-open, $\beta$-open) sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

By the definitions stated above, we have the following diagram:

$$
\begin{align*}
\text{normality} & \Rightarrow \alpha\text{-normality} \Rightarrow p\text{-normality} \Rightarrow \beta\text{-normality} \\
\downarrow \\
\xi\text{-normality}
\end{align*}
$$

Where none of the implications is reversible as can be seen from the following examples:

4.2 Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. The pair of disjoint closed subsets of $X$ are $A = \{a\}$ and $B = \{c\}$. Taking $\beta$-open sets, $U = \{a, b\}$ and $V = \{c, d\}$ such that $A \subseteq U$ and $B \subseteq V$. Hence the space $X$ is $\beta$-normal. But the space $X$ is neither $p$-normal nor $\alpha$-normal, since the sets $U$ and $V$ are neither $p$-open nor $\alpha$-open.

4.3 Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{b, d\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. The pair of disjoint closed subsets of $X$ are $A = \{a\}$ and $B = \{c\}$. Taking $p$-open sets, $U = \{a, b\}$ and $V = \{c, d\}$ such that $A \subseteq U$ and $B \subseteq V$. Hence the space $X$ is $p$-normal as well as $\beta$-normal, since every $p$-open sets are $\beta$-open. But the space $X$ is neither normal nor $\alpha$-normal, since the sets $U$ and $V$ are neither open nor $\alpha$-open.

4.4 Theorem. The following properties are equivalent for a space $X$:

(a) $X$ is $\xi$-normal.
(b) For each pair of disjoint closed sets $A$, $B$ of $X$, there exist disjoint $\xi$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
(c) For each closed set $A$ and any open set $V$ containing $A$, there exists a $\xi$-open set $U$ such that $A \subseteq U \subseteq \xi Cl(U) \subseteq V$.
(d) For each closed set $A$ and any open set $B$ containing $A$, there exists a $\xi$-open set $U$ such that $A \subseteq U \subseteq \xi Cl(U) \subseteq \int(B)$.
(e) For each closed set $A$ and any open set $B$ containing $A$, there exists a $\xi$-open set $G$ such that $A \subseteq \xi Cl(G) \subseteq \int(B)$.
(f) For each closed set $A$ and any open set $B$ containing $A$, there exists a $\xi$-open set $U$ such that $\xi Cl(A) \subseteq U \subseteq \xi Cl(U) \subseteq B$.
(g) For each closed set $A$ and any open set $B$ containing $A$, there exists a $\xi$-open set $G$ such that $\xi Cl(A) \subseteq G \subseteq \xi Cl(G) \subseteq B$.

Proof. (a) $\Rightarrow$ (b). This proof is obvious since every $\xi$-open set is $\xi$-open.

(b) $\Rightarrow$ (c). Let $A$ be a closed set and let $V$ be an open set containing $A$. Since $A$ and $V - A$ are disjoint closed sets of $X$, there exist $\xi$-open sets $U$ and $W$ of $X$ such that $A \subseteq U$ and $X - V \subseteq W$ and $U \cap W = \emptyset$. By Lemma 2.9, we get $X - V \subseteq \xi \int(W)$. Since $U \cap \xi \int(W) = \emptyset$, we have $\xi Cl(U) \cap \xi \int(W) = \emptyset$ and hence $\xi Cl(U) \subseteq X - \xi \int(W) \subseteq V$. Therefore, we obtain $A \subseteq U \subseteq \xi Cl(U)$.

(c) $\Rightarrow$ (a). Let $A$ and $B$ be the disjoint closed sets of $X$. Since $X - B$ is an open set containing $A$, there exists a $\xi$-open set $G$ such that $A \subseteq G \subseteq \xi Cl(G) \subseteq X - B$. By Lemma 2.9, we have $A \subseteq \xi \int(G)$. Put $U = \xi \int(G)$ and $V = X - \xi Cl(G)$. Then $U$ and $V$ are disjoint $\xi$-open sets such that $A \subseteq U$ and $B \subseteq V$. Therefore $X$ is $\xi$-normal.

Since every $\xi$-open set is $\xi$-open and every closed (resp. open) set is $\xi$-closed (resp. $\xi$-open), it is obvious that (d) $\Rightarrow$ (e) and (f) $\Rightarrow$ (g) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d). Let $A$ be a closed set of $X$ and let $B$ be a $\xi$-open set such that $A \subseteq B$. Since $B$ is $\xi$-open and $A \subseteq \int(B)$ by Lemma 2.9. Therefore by (c), there exists a $\xi$-open set $U$ such that $A \subseteq U \subseteq \xi Cl(U) \subseteq \int(B)$.

(e) $\Rightarrow$ (d). Let $A$ be a closed set of $X$ and let $B$ be a $\xi$-open set such that $A \subseteq B$. Then there exists a $\xi$-open set $G$ such that $A \subseteq G \subseteq \xi Cl(G) \subseteq \int(B)$ by Lemma 2.9. Since $G$ is $\xi$-open, $A \subseteq \int(G)$. Put $U = \int(G)$, then $U$ is $\xi$-open and $A \subseteq U \subseteq \xi Cl(U) \subseteq \int(B)$.

(g) $\Rightarrow$ (f). Let $A$ be a $\xi$-closed set of $X$ and let $B$ be an open set such that $A \subseteq B$. Then there exist a $\xi$-open set $G$ such that $\xi Cl(A) \subseteq G \subseteq \xi Cl(G) \subseteq B$. Since $G$ is $\xi$-open and the closed set $\xi Cl(A) \subseteq G$, we have $\xi Cl(A) \subseteq \int(G)$ by Lemma 2.9. Put $U = \int(G)$. Then, $U$ is $\xi$-open and $\xi Cl(A) \subseteq U \subseteq \int(B)$.

4.5 Theorem. If $f : X \to Y$ is continuous $\xi$-closed surjection and $X$ is normal, then $Y$ is $\xi$-normal.

Proof. Let $A$ and $B$ be the disjoint closed sets of $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of $X$ since $f$ is continuous. Since $X$ is normal, there exists disjoint open sets $U$ and $V$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By Proposition 3.6, there exist $\xi$-open sets $G$ and $H$ of $Y$ such that $A \subseteq G$, $B \subseteq H$ and $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and hence $G \cap H = \emptyset$. It follows from Theorem 4.4 that $Y$ is $\xi$-normal.

4.6 Theorem. If $f : X \to Y$ is continuous $\xi$-closed surjection and $X$ is $\xi$-normal, then $Y$ is $\xi$-normal.
Proof. Let A and B the disjoint closed sets of Y. Then \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint closed sets of X. Since X is \( \xi \)-normal, there exist disjoint \( \xi \)-open sets U and V such that \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \). Since \( f \) is \( \xi \)-g\( \xi \)-closed, by Proposition 3.6, there exist \( \xi \)-open sets G and H of Y such that \( A \subseteq G, B \subseteq H, f^{-1}(G) \subseteq U \) and \( f^{-1}(H) \subseteq V \). Since U and V are disjoint, we have \( G \cap H = \emptyset \). This shows that Y is \( \xi \)-normal.

5. \( \xi \)-Regular Spaces

5.1 Definition. A space X is said to be \( \xi \)-regular [10] (resp. \( \alpha \)-regular [1]) if for each closed set F of X, and each point \( x \in X - F \), there exist disjoint \( \xi \)-open (resp. \( \alpha \)-open) set U, V such that \( F \subseteq U \) and \( x \in V \).

5.2 Remark. It is obvious that every \( \alpha \)-regular space is \( \xi \)-regular but not conversely.

5.3 Lemma. The following properties are equivalent for a space X:

(a) X is \( \xi \)-regular.

(b) For each \( x \in X \) and each open set U of X containing x, there exists \( V \subseteq \xi O(X) \) such that \( x \in V \subseteq \xi Cl(V) \subseteq U \).

(c) For each closed set F of X, \( \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} = F \).

(d) For each subset A of X and each closed set U of X such that \( A \cap U \neq \emptyset \), there exists \( V \subseteq \xi O(X) \) such that \( A \cap V \neq \emptyset \) and \( \xi Cl(V) \subseteq U \).

(e) For each non-empty subset A of X and each closed subset F of X such that \( A \cap F = \emptyset \), there exist \( V, W \subseteq \xi O(X) \) such that \( A \cap W \neq \emptyset \), \( F \subseteq V \) and \( V \cap W \neq \emptyset \).

Proof.

(a) \( \Rightarrow \) (b). Let U be an open set containing x, then \( X - U \) is closed in X and \( x \notin X - U \). By (a), there exist \( W, V \subseteq \xi O(X) \) such that \( x \in V \subseteq \xi Cl(V) \subseteq U \).

(b) \( \Rightarrow \) (c). Let F be a closed set of X. If \( F \subseteq V \), then by Lemma 2.2 (iii), \( \xi Cl(F) \subseteq \xi Cl(V) \) which gives \( F \subseteq \xi Cl(V) \) as \( F \subseteq \xi Cl(F) \). Therefore, \( \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} \supseteq F \).

Conversely, let \( x \notin F \). Then \( X - F \) is an open set containing x. By (b), there exists \( U \subseteq \xi O(X) \) such that \( x \in U \subseteq \xi Cl(U) \subseteq X - F \). Put \( V = X - \xi Cl(U) \). By Lemma 2.2, \( F \subseteq V \subseteq \xi O(X) \) and \( x \notin \xi Cl(V) \). This implies that \( \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} \subseteq F \).

Hence \( \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} = F \).

(c) \( \Rightarrow \) (d). Let A be a subset of X and let U be open in X such that \( A \cap U \neq \emptyset \). Let \( x \in A \cap U \), then \( X - U \) is a closed set not containing x. By (c), there exists \( W \subseteq \xi O(X) \) such that \( X - U \subseteq W \) and \( x \notin \xi Cl(W) \). Put \( V = X - \xi Cl(W) \). Then \( V \subseteq X - W \). Also \( x \in V \cap A \). By using Lemma 2.2, we obtain \( V \subseteq \xi O(X) \) and \( \xi Cl(V) \subseteq \xi Cl(X - W) \).

(d) \( \Rightarrow \) (e). Let A be a subset of X and let F be a closed set in X such that \( A \cap F = \emptyset \), where \( A \neq \emptyset \). Since X - F is open in X and \( A \neq \emptyset \), by (d), there exists \( V \subseteq \xi O(X) \) such that \( A \cap V \neq \emptyset \) and \( \xi Cl(V) \subset X - F \). Put \( W = X - \xi Cl(V) \), then \( F \subseteq W \). Also, \( V \cap W = \emptyset \). By Lemma 2.2, \( W \subseteq \xi O(X) \).

(e) \( \Rightarrow \) (a). This is obvious.

5.4 Theorem. The following properties are equivalent for a space X:

(a) X is \( \xi \)-regular.

(b) For each closed set F and each point \( x \in X - F \), there exists \( U \subseteq \xi O(X) \) and a \( g\xi \)-open set V such that \( x \in U \) and \( F \subseteq V \) and \( U \cap V = \emptyset \).

(c) For each subset A of X and each closed set F such that \( A \cap F = \emptyset \), there exist \( U \subseteq \xi O(X) \) and a \( g\xi \)-open set V such that \( A \subseteq U \) and \( F \subseteq V \).

(d) For each closed set F of X, \( F \cap \subseteq \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} \).

Proof.

(a) \( \Rightarrow \) (b). The proof is obvious since every \( \xi \)-open set is \( g\xi \)-open.

(b) \( \Rightarrow \) (c). Let A be a subset of X and let F be a closed set in X such that \( A \cap F = \emptyset \). For a point \( x \in A \), \( x \notin X - F \) and hence there exists \( U \subseteq \xi O(X) \) and a \( g\xi \)-open set V such that \( x \in U \) and \( F \subseteq V \) and \( U \cap V = \emptyset \).

(c) \( \Rightarrow \) (a). Let F be a closed set and let \( x \in X - F \). Then, \( \{ x \} \cap \subseteq \emptyset \) and there exist \( U \subseteq \xi O(X) \) and a \( g\xi \)-open set W such that \( x \in U \), \( F \subseteq W \) and \( U \cap W = \emptyset \). Put \( V = \xi Int(W) \), then by Lemma 2.9, we have \( F \subseteq V \subseteq \xi O(X) \) and \( \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} = F \). Therefore X is \( \xi \)-regular.

(a) \( \Rightarrow \) (d). For a closed set F of X, by Lemma 5.3, we obtain

\( F \subseteq \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} \subseteq \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} = F \).

Therefore, \( F = \cap \{ \xi Cl(V) / F \subseteq V \subseteq \xi O(X) \} = \xi g\xi -open \).

(d) \( \Rightarrow \) (a). Let F be a closed set of X and x \( x \in X - F \). By (d), there exists a \( g\xi \)-open set W of X such that \( F \subseteq W \) and \( x \notin X - \xi Cl(W) \). Since F is closed, \( F \subseteq \xi Int(W) \) by Lemma 2.9. Put \( V = \xi Int(W) \), then \( F \subseteq V \subseteq \xi O(X) \). Since \( x \in X - \xi Cl(W) \), \( x \notin X - \xi Cl(V) \). Put \( U = X - \xi Cl(V) \), then \( x \in U \), \( U \subseteq \xi O(X) \) and \( U \cap V = \emptyset \). This shows that X is \( \xi \)-regular.

5.5 Definition. A function \( f : X \to Y \) is said to be \( \xi \)-open [2] if for each open set U of X, \( f(U) \subseteq \xi O(Y) \).
5.6 Theorem. If $f: X \to Y$ is a continuous $\xi$-open $g\xi$-closed surjection and $X$ is regular, then $Y$ is $\xi$-regular.

Proof. Let $y \in Y$ and let $V$ be an open set of $Y$ containing $y$. Let $x$ be a point of $X$ such that $y = f(x)$. By the regularity of $X$, there exists an open set $U$ of $X$ such that $x \in U \subset Cl(U) \subset f^{-1}(V)$. We have $y \in f(U) \subset f(Cl(U)) \subset V$. Since $f$ is $\xi$-open and $g\xi$-closed, $f(U) \in \xi O(Y)$ and $f(Cl(U))$ is $g\xi$-closed in $Y$. So, we obtain, $y \in f(U) \subset Cl(f(Cl(U))) \subset V$. It follows from Lemma 5.4 that $Y$ is $\xi$-regular.

5.7 Definition. A function $f: X \to Y$ is said to be pre $\xi$-open if for each $\xi$-open set $U$ of $X$, $f(U) \in \xi O(Y)$.

5.8 Theorem. If $f: X \to Y$ is a continuous pre $\xi$-open $g\xi$-closed surjection and $X$ is $\xi$-regular, then $Y$ is $\xi$-regular.

Proof. Let $F$ be any closed set of $Y$ and $y \in Y - F$. Then $f^{-1}(Y) \cap f^{-1}(F) = \emptyset$ and $f^{-1}(F)$ is closed in $X$. Since $X$ is $\xi$-regular, for a point $x \in f^{-1}(y)$, there exist $U, V \in \xi O(X)$ such that $x \in U, f^{-1}(F) \subset V$ and $U \cap V = \emptyset$. Since $F$ is closed in $Y$, by Proposition 3.6, there exists $W \in \xi O(Y)$ such that $F \subset W$ and $f^{-1}(W) \subset V$. Since $f$ pre $\xi$-open, we have $y = f(x) \in f(U)$ and $f(U) \in \xi O(Y)$. Since $U \cap V = \emptyset$, $f^{-1}(W) \cap U = \emptyset$ and hence $W \cap f(U) = \emptyset$. This shows that $Y$ is $\xi$-regular.

6. Conclusion

We introduced a weaker version of normality called $\xi$-normality in topological spaces. We gave some characterizations and preservation theorems of $\xi$-normal and $\xi$-regular spaces. Some counterexamples were given and some basic properties were presented. The relationships among normal, $\alpha$-normal, $p$-normal, $\beta$-normal, and $\xi$-normal are investigated.

References