Portfolio Selection with fuzzy return without target values

Yves Tinda Mangongo[1], **Christian Tassak Deffo**[2], **Louis Aimé Fono**[3], **Joseph Désiré Kyemba Bukweli**[4] **and Justin Dupar Busili Kampempe**[5]

[1, 4, 5]Department of Mathematics and Computer Science
University of Kinshasa, Kinshasa, D.R.Congo
Email: ymangongo@gmail.com; ymangongo@aims.ac.tz
Email: desbuk@gmail.com
Email: justin-dupar.kampempe@unikin.ac.cd

[2]Department of Mathematics
University of Yaoundé I, Yaoundé, Cameroon
Email: christade_24@yahoo.fr

[3]Department of Mathematics and Computer Science
University of Douala, Douala, Cameroon
Email: lfono2000@gmail.com

Abstract

The aim of this paper is to solve the question of portfolio selection with fuzzy returns by means of parameters and second order dominance of fuzzy variables. We have proposed two new multiobjectives models for selecting the best portfolio, based on parameters and on second order dominance of fuzzy variables, respectively. In addition, characterizations of second order dominance for trapezoidal and triangular fuzzy numbers are introduced.

Key Words: Fuzzy variable, Second order dominance, Credibility measure, Set of the best portfolios, Multiobjective problem.

Introduction

The portfolio selection theory involves a selection of a combination of assets under the constraints of the investor objectives. Given a finite number of assets, the main question in portfolio selection theory is, how to invest a certain capital to that finite number of assets; such that the future return obtained from that investment has a maximum profit and less risk. Markowitz (1952), the pioneer founder of portfolio selection theory, provided a mathematical foundation for modern portfolio selection theory as a mathematical problem. Markowitz considers that the future return for a given portfolio is a random variable. With this consideration, he proposed the mean-variance model, which opened the door for mathematical analysis of the portfolio selection problem Sadefo et al. (2012a).

The key principle of the mean-variance model is to use the expected return of a portfolio as the investment return, and to use the variance of a portfolio as the investment risk Carlsson et al. (2001). The model was formulated as shown in Markowitz (1952). Consider a portfolio with n different assets where asset number i gives the return R_i. Let μ_i and σ_i^2 be the corresponding mean and variance respectively, and let $\sigma_{i,j}$ be the covariance between R_i and R_j. Suppose the relative amount of the value of the portfolio invested in asset i is x_i and if R is the return of the whole portfolio then:

$$\begin{cases} \max E[R] = \sum_{i=1}^{n} \mu_i x_i, \\ \text{subject to,} \\ \sigma^2 = Var[R] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j} x_i x_j, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0\,, \;\; i = 1, \ldots, n. \end{cases} \tag{1}$$

However, sometimes we encounter situations where future return does not describe a probability distribution. In such cases, we can not use a random variable to describe a future return for a given portfolio. For instance, if there is not enough historical data, it is more reasonable to assume that a future return is a fuzzy variable Li et al. (2010). In addition, the security return is very sensitive to economic, environmental, political, social and human's psychological factors Sadefo et al. (2012a). This justifies the inefficient information or the incomplete information that an investor can receive from the real world. To reflect this vagueness or incompleteness due to the lack of data, the fuzzy set theory introduced by Zadeh (1965), through fuzzy variable will be used for modelling future return.

As in probabilistic approach, where the expected value of a random variable and its variance have been used to characterize respectively the expected future return and the investment risk. Scholars such as Huang (2008), Li et al. (2010) and Sadefo et al. (2012a) used the same parameters for a fuzzy variable in fuzzy approach. To define these parameters, Liu and Liu (2002) introduced credibility measure in the new theory of uncertainty. The main advantage of this measure is that, it is self-dual. The self-duality property helps to make decision results consistent with the laws of contradiction and excluded middle Sadefo et al. (2012a).

Consequently, so many models for portfolio selection with fuzzy return have been done. We have Huang (2008), who proposed Mean-semivariance models for portfolio selection with fuzzy return; Li et al. (2010), who proposed Mean-variance-skewness model for portfolio selection with fuzzy return; and Sadefo et al. (2012a), who also proposed Mean-variance-skewness-kurtosis model for portfolio selection with fuzzy return. All these models on parameters are based on chosen target values for constraints. In addition, Tassak (2016), Tassak et al. (2016) and Sadefo et al. (2012b) studied and characterized three dominances on fuzzy variables and used the first order dominance to solve portfolio question in fuzzy case. Following this way, our modest contribution is to solve the question of portfolio selection with fuzzy return: multiobjective model (model based on parameters without target values) and multiobjective model based on the second order dominance of fuzzy variables. Likewise, two propositions on characterization of the second order dominance on trapezoidal and triangular fuzzy numbers have been newly introduced.

The paper is organised as follows. In section 1, we review some basics concepts about fuzzy set theory and we end the section by defining second order dominance of fuzzy variables. The second section discusses about the models of our research, and we are going to display two numerical examples for showing the effectiveness of our proposed multiobjective models. The third section gives the new characterisations of the second order dominance for trapezoidal and triangular fuzzy numbers. Finally, we are going to end the paper by some concluding remarks in which we propose one open question for the future work.

1 Preliminaries

In this section, we provide some definitions and previous results that will be used in this paper. Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of Θ (the largest $\sigma-$algebra over Θ). Each element in $\mathcal{P}(\Theta)$ is called an event. The set function $Cr : \mathcal{P}(\Theta) \to [0,1]$ is called a credibility measure if $Cr\{\Theta\} = 1$ (normality axiom), $Cr\{A\} \le Cr\{B\}$ whenever $A \subseteq B$ (monotonicity axiom), $Cr\{A\} + Cr\{A^c\} = 1$ for any event $A \in \mathcal{P}(\Theta)$ (self-duality axiom) and $Cr\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $Cr\{A_i\} < 0.5$ (maximality axiom). The value of $Cr\{A\}$ represents the level that an event A occurs. The meaning of maximality axiom is followed, there is not uncertainty in the outcome of an event if its credibility measure is 1 or 0 because we are sure that the event occurs or not. The triplet $(\Theta, \mathcal{P}(\Theta), Cr)$ is called a credibility space. A fuzzy variable is defined as a (measurable) function from a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ to the set of real numbers (see Huang (2010) and Gupta et al. (2014)). If ξ is a fuzzy variable defined on a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$; then its membership function is derived from the credibility measure by:

$$\mu(t) = (2Cr\{\xi = t\}) \wedge 1, \ \ t \in \mathbb{R}; \tag{2}$$

which, for a given fuzzy number ξ and for any $x \in \mathbb{R}$, $\mu(x)$ represents the grade of x belongs to ξ.

Definition 1 *Tassak et al. (2016). A fuzzy number is a fuzzy variable satisfying the following four axioms, $\exists a,b,c,d \in \mathbb{R}$ with $a < b < c < d$ such that: $\forall r \in [b,c], \mu(r) = 1$; $\forall r \notin]a,d[, \mu(r) = 0$; μ is increasing on $[a,b]$ and decreasing on $[c,d]$; and μ is upper semi-continuous.*

If ξ is a fuzzy variable, then ξ is said to be *trapezoidal fuzzy variable* if its membership function is fully determined by the quadruplet (a,b,c,d) with $a < b \le c \le d$, real numbers and defined by

$$\mu_1(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \le x < b, \\ 1, & \text{if } b \le x \le c, \\ \frac{d-x}{d-c}, & \text{if } c \le x < d, \\ 0, & \text{othewise}; \end{cases}$$

If $b = c$, then the trapezoidal fuzzy variable $\xi = (a,b,c,d)$ becomes the triangular fuzzy variable $\xi = (a,b,d)$.

If the membership function of a fuzzy variable ξ is known, now the question is how to determine the credibility value of a fuzzy event, Gupta et al. (2014)? In order to measure a fuzzy event, Liu and Liu (2002) introduced a credibility measure as follow. If ξ is a fuzzy variable with membership function μ, then for any subset A of real numbers, we have,

$$Cr\{\xi \in A\} = \frac{1}{2}\left(\sup_{t \in A} \mu(t) + 1 - \sup_{t \in A^c} \mu(t)\right). \tag{3}$$

The Equality (3), can be formulated as

$$Cr\{\xi \in A\} = \frac{1}{2}\left(Pos(\{\xi \in A\}) + Nec(\{\xi \in A\})\right), \tag{4}$$

with possibility and necessity measures of A defined by L. Zadeh in Bouchon-Meunier (1993) respectively by $Pos(\{\xi \in A\}) = \sup_{t \in A} \mu(t)$ and $Nec(\{\xi \in A\}) = 1 - \sup_{t \in A^c} \mu(t)$. But neither, of these measures are self-dual. That reason also justified the introduction of credibility measure by Liu and Liu (2002).

Definition 2 (Credibility Distribution) *The credibility distribution, $\Phi : \mathbb{R} \to [0,1]$ of a fuzzy variable is defined by*

$$\Phi(t) = Cr\{\xi \le t\}. \tag{5}$$

Huang (2010) demonstrated that if ξ is a fuzzy variable with membership function μ, then the credibility distribution of ξ is

$$\Phi(t) = \frac{1}{2}\left(\sup_{z\le t}\mu(z) + 1 - \sup_{z>t}\mu(z)\right), \ \forall t \in \mathbb{R}. \tag{6}$$

Definition 3 *Let ξ be a fuzzy variable. One defines, the expected value of ξ by*

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \ge r\}\,dr - \int_{-\infty}^{0} Cr\{\xi \le r\}\,dr,$$

provided that at least one of the two integrals is finite.

One can prove that if $\xi = (a,b,c,d)$ is a trapezoidal fuzzy variable, then the expected value of ξ, is given by

$$E[\xi] = \frac{a+b+c+d}{4}. \tag{7}$$

Moreover if $\xi = (a,b,c)$ is a triangular fuzzy variable, then the expected value of ξ, is deduced from (7):

$$E[\xi] = \frac{a+2b+c}{4}. \tag{8}$$

Liu (2002) introduce the variance a fuzzy variable ξ with finite expected value e by:

$$V[\xi] = E[(\xi - e)^2]. \tag{9}$$

That is the variance is simply the expected value of $(\xi - e)^2$. Since $(\xi - e)^2$ is nonnegative uncertain variable, we also have

$$V[\xi] = \int_0^{+\infty} Cr\left\{(\xi - e)^2 \ge r\right\}dr. \tag{10}$$

The skewness of a fuzzy variable ξ with a finite expected value e is defined by

$$S[\xi] = E[(\xi - e)^3]. \tag{11}$$

Sadefo et al. (2012a) proved that if ξ has a symmetric membership function, then $S[\xi] = 0$. In the sequel they introduced the Kurtosis of a fuzzy variable with a finite expected value e, denoted $K[\xi]$ and given by:

$$K[\xi] = E[(\xi - e)^4]. \tag{12}$$

Recently, Tassak et al. (2016) introduced the second order credibility dominance as the binary relation between fuzzy variables denoted by $\succeq_2$ and defined as follows: if ξ_1 and ξ_2 are two fuzzy variables with Φ_1 and Φ_2 their corresponding credibility distribution functions, respectively, then

$$\xi_1 \succeq_2 \xi_2 \ \text{ whenever } \ \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \ge 0. \tag{13}$$

This integral represents a balance of areas between the curves of Φ_1 and Φ_2.

Therefore, one can deduce the strict dominance relations by: $\xi_1 \succ_2 \xi_2$ if and only if

$$\begin{cases} \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \ge 0, \\ \exists t_0 \in \mathbb{R}, \int_{-\infty}^{t_0} [\Phi_2(r) - \Phi_1(r)]dr > 0. \end{cases} \tag{14}$$

Sometimes it is quite difficult to use a definition to characterize a given problem. That is the reason Tassak et al. (2016) provided the following theorem, as a characterization of the second order dominance.

Theorem 1 (Characterization of the Second Order Dominance) *Let ξ_1 and ξ_2 be two fuzzy variables with a finite number of crossing points $\{t_{01},\dots,t_{0k}\}$ (ordered so increasing) such that $t_{01} > \min\{\inf\{t : \Phi_1(t) > 0\}, \inf\{t : \Phi_2(t) > 0\}\}$. Let Φ_1 and Φ_2 their respective absolutely continuous credibility distributions. Then $\xi_1 \succ_2 \xi_2$ if and only if*

1. $\forall i \in \{1,2,\dots,k\}, \int_{-\infty}^{t_{0i}}[\Phi_2(r)-\Phi_1(r)]dr \geq 0;$
2. *And either one of the following holds:*
   (a) $\int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr = 0$ *and* $\exists t_{0h} \in \{t_{01},\dots,t_{0k}\}, \int_{-\infty}^{t_{0h}}[\Phi_2(r)-\Phi_1(r)]dr > 0,$
   (b) $\int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr > 0.$

We have two types of crossing points:

1. A crossing point of type I, that is an upper bound of an interval of coincidence (an interval of coincidence, IC, is a half-open interval $[a,b)$ where $\Phi_1(r) = \Phi_2(r)$, $\forall r \in [a,b)$).
2. A crossing point of type II is a point where the two curves of the credibility distribution functions of two fuzzy variables intersect. For more information, we refer to Tassak et al. (2016).

In the next section, we propose two multiobjective models for portfolio selection with fuzzy return. The first model based on parameters of a fuzzy variable and the second one based on second order dominance of fuzzy variables.

2 Models

Given a certain portfolio with $n \geq 2$ assets, $n \in \mathbb{N}$, the fuzzy return ξ_i for asset i is formulated as follows. Let x_i be the proportion of the capital invested in security i. The future return ξ_i is defined as $\xi_i = (p_i^{'} + d_i - p_i)/p_i$ respectively for $i = 1,2,\dots,n$, where d_i is the estimated dividend of the security i during the coming year, which is unknown at present; p_i is the closing price of the security i at the present; finally $p_i^{'}$ is the estimated closing price of the security i in the next year, which is also unknown at present. So the future return for the whole portfolio is also a fuzzy variable, given by

$$\xi = \sum_{i=1}^{n} x_i \xi_i.$$

2.1 First approach: Model based on parameters of a fuzzy variable.

Many authors such as Huang (2008), Li et al. (2010), Sadefo et al. (2012a), made models for portfolio selection with fuzzy returns by considering the parameters such as expected value, variance, semivariance, skewness, kurtosis of a fuzzy variable. But all those models used threshold (target) values for each constraint which can be chosen arbitrarily. In addition, sometimes to require a target value for a given parameter to an investor can be difficult. To alleviate this shortcoming, we propose a new multiobjective problem (model) based on parameters for portfolio selection with fuzzy return; as our first approach. This model maximizes the odd moment, such as expected value and skewness, and minimizes the even moment, such as variance and kurtosis. The model is a favourite approach for investors who don't fear risk; because sometimes it is suitable for an investor to optimize in the same time mean, variance, skewness and kurtosis.

Formally we have the model as shown below;

$$\begin{cases} \max E[\xi], \\ \max S[\xi], \\ \min V[\xi], \\ \min K[\xi], \\ \text{subject to,} \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0\ ,\ i = 1,2,\ldots,n. \end{cases} \tag{15}$$

The feasible set $\Omega \subseteq \mathbb{R}^n$ of Problem (15), is defined as: $\Omega = \left\{ x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = 1, x_i \in [0,1], i = 1,\ldots,n \right\}$, which is non empty. The first objective of our model is to maximize the expected future return, the second objective is to maximise the asymmetry of our future return, which is described by the skewness, the third objective is to minimize the variance of the future return, the variance which is a measure of the investment risk, and finally, the last objective is to minimize the kurtosis of our future return, which indicates the fat-tails or the thin-tails of the distribution of our future return. The first constraint assures that all the capital will be invested in n assets, and the last constraints assure that the short-selling is not allowed.

2.2 Second approach: Model based on the Second Order Dominance

To give our second approach, let us define first the set of the best portfolio.

Definition 4

Let $A = (\xi_i)_{1 \leq i \leq n}$ be a family of n triangular fuzzy assets' returns. A portfolio return ξ associated with A is a linear combination of the n assets' returns defined by $\xi = \sum_{i=1}^{n} x_i \xi_i$ where x_i represents the proportion of the capital invested in asset i. So, we have $x_i \in [0,1]$ and $\sum_{i=1}^{n} x_i = 1$, that is all the capital should be invested on n assets. Hence we have the set $\mathcal{P}$, of portfolios associated with A defined as $\mathcal{P} = \left\{ \xi = \sum_{i=1}^{n} x_i \xi_i,\ x_i \in [0,1], \sum_{i=1}^{n} x_i = 1 \text{ and } \xi_i \in A \right\}$.

Now, the question is, how to determine the set of the best portfolios of $\mathcal{P}$ based on second order dominance, denoted by $\mathcal{B}_{\succ_2}$, and defined as $\mathcal{B}_{\succ_2} = \{\xi \in \mathcal{P}, \forall \eta \in \mathcal{P},\ \xi \succ_2 \eta\}$, with $\eta = \sum_{i=1}^{n} y_i \xi_i$, $\forall y_i \in [0,1]$ and $\sum_{i=1}^{n} y_i = 1$.

If $\xi = (a,b,c,d)$ is a trapezoidal fuzzy variable. Its credibility distribution function Φ is given by: $\forall r \in \mathbb{R}$,

$$\Phi(r) = \begin{cases} 0 & \text{if } r < a, \\ \frac{1}{2}(\frac{r-a}{b-a}) & \text{if } a \leq r < b, \\ \frac{1}{2} & \text{if } b \leq r < c, \\ 1 - \frac{1}{2}(\frac{r-d}{c-d}) & \text{if } c \leq r < d, \\ 1 & \text{if } d \leq r \end{cases} . \tag{16}$$

It has been proved by means of crossing points that two credibility distribution functions' curves have the same position as their respective membership functions' curves. For the case of a finite combination of trapezoidal fuzzy variables (portfolio), an implementation of the set of best portfolios can be done in the following way:

With a portfolio of n assets described by trapezoidal fuzzy variables, best portfolios with respect to the second order dominance can be implemented as follows:

$$\begin{cases} \max a_1x_1 + ... + a_nx_n \\ \max c_1x_1 + ... + c_nx_n \\ \min \frac{1}{(b_1-a_1)x_1+...+(b_n-a_n)x_n} \\ \min \frac{1}{(d_1-c_1)x_1+...+(d_n-c_n)x_n} \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \in [0,1], \forall i \in \{1,\dots,n\} \end{cases} . \tag{17}$$

Model (17), constitute the second approach of this paper. In the last subsection, we are going to do two numerical examples to show the effectiveness of our proposed models.

2.3 Numerical examples

In order to solve our proposed models, Matlab through genetic algorithm with fuzzy simulation is used. Genetic algorithm and fuzzy simulation can be found in Huang (2008). The data that we have used, have been introduced and used the first time by Huang (2008), and used by others scholars such as Li et al. (2010), Sadefo et al. (2012a). The motivation of using those data is that, we want to do comparison with the existed results from the others scholars. Those data contain seven triangular securities returns shown in Table 1.

Table 1: Data

security i	Fuzzy return	security i	Fuzzy return
1	$\xi_1 = (-0.3, 1.8, 2.3)$	5	$\xi_5 = (-0.7, 2.4, 2.7)$
2	$\xi_2 = (-0.4, 2.0, 2.2)$	6	$\xi_6 = (-0.8, 2.5, 3.0)$
3	$\xi_3 = (-0.5, 1.9, 2.7)$	7	$\xi_7 = (-0.6, 1.8, 3.0)$
4	$\xi_4 = (-0.6, 2.2, 2.8)$		

2.3.1 Some best portfolios : implementation with two approaches and interpretations

The first approach is Model (15) describes in Subsection 2.1 of the present paper. Using the seven triangular fuzzy assets returns of Table 1, Model (15), becomes;

$$\begin{cases} \max E[\xi], \\ \max S[\xi], \\ \min V[\xi], \\ \min K[\xi], \\ \text{subject to,} \\ \sum_{i=1}^{7} x_i = 1, \\ x_i \geq 0 \, , \, i = 1, 2, \dots, 7. \end{cases} \tag{18}$$

The feasible set $\Omega \subseteq \mathbb{R}^7$ of Problem (18), is defined as: $\Omega = \left\{ x \in \mathbb{R}^7 | \sum_{i=1}^{7} x_i = 1, x_i \in [0,1], i = 1, \dots, 7 \right\},$

which is non-empty. Problem (18) is equivalent to

$$\begin{cases} \max \dfrac{f_1(x)+2f_2(x)+f_3(x)}{4} \\ \max \dfrac{(f_3(x)-f_1(x))^2}{32}(f_3(x)+f_1(x)-2f_2(x)) \\ \min \dfrac{33\alpha^3+21\alpha^2\beta+11\alpha\beta^2-\beta^3}{384\alpha} \\ \min \dfrac{253\alpha^5+395\alpha^4\beta+17\alpha\beta^4+290\alpha^3\beta^2+70\alpha^2\beta^3-\beta^5}{10240\alpha} \\ \text{subject to} \\ \sum_{i=1}^{7} x_i = 1 \\ x_i \geq 0\ ,\ i=1,2,\ldots,7 \end{cases} ; \tag{19}$$

with $\alpha = \max\{f_2(x)-f_1(x), f_3(x)-f_2(x)\}, \beta = \min\{f_2(x)-f_1(x), f_3(x)-f_2(x)\}$. Likewise, $f_1(x) = -0.3x_1 - 0.4x_2 - 0.5x_3 - 0.6x_4 - 0.7x_5 - 0.8x_6 - 0.6x_7, f_2(x) = 1.8x_1 + 2.0x_2 + 1.9x_3 + 2.2x_4 + 2.4x_5 + 2.5x_6 + 1.8x_7, f_3(x) = 2.3x_1 + 2.2x_2 + 2.7x_3 + 2.8x_4 + 2.7x_5 + 3.0x_6 + 3.0x_7$. After writing this nonlinear multiobjective problem, we have used Matlab to solve it. After running the codes, we obtained the following portfolios returns chosen among several others.

$$\xi^{(1)} = 0.4234\xi_1 + 0.0085\xi_2 + 0.0054\xi_3 + 0.0032\xi_4 + 0.1576\xi_5 + 0.3944\xi_6 + 0.0076\xi_7,$$
$$\xi^{(2)} = 0.8705\xi_1 + 0.0076\xi_2 + 0.0201\xi_3 + 0.0016\xi_4 + 0.0264\xi_5 + 0.0726\xi_6 + 0.0011\xi_7.$$

An investor who share its capital as described on portofolios $\xi^{(1)}$ and $\xi^{(2)}$, expect to get values as described in Table 2.

Now we are going to do a numerical example for our second approach, Model (17).

Using the seven triangular fuzzy assets returns of Table 1, Model (17), becomes:

$$\begin{cases} \max -0.3x_1 - 0.4x_2 - 0.5x_3 - 0.6x_4 - 0.7x_5 - 0.8x_6 - 0.6x_7, \\ \min \dfrac{1}{2.1x_1 + 2.4x_2 + 2.4x_3 + 2.8x_4 + 3.1x_5 + 3.3x_6 + 2.4x_7}, \\ \min \dfrac{1}{0.5x_1 + 0.2x_2 + 0.8x_3 + 0.6x_4 + 0.3x_5 + 0.5x_6 + 1.2x_7}, \\ \text{subject to,} \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 1, \\ x_i \in [0,1]\ ,\ i = 1,\ldots,7. \end{cases} \tag{20}$$

To solve Problem (20), Matlab via genetic algorithm has been used. After running the codes, we obtained the following two portfolios chosen among several others; that is we obtain the two elements of $\mathcal{B}_{>_2}$.

$$\begin{aligned} \xi^{(3)} &= 1\xi_1 + 0\xi_2 + 0\xi_3 + 0\xi_4 + 0\xi_5 + 0\xi_6 + 0\xi_7 = \xi_1 \\ &= (-0.3, 1.8, 2.3), \\ \xi^{(4)} &= 0.9129\xi_1 + 0.0189\xi_2 + 0.0028\xi_3 + 0.0013\xi_4 + 0.0168\xi_5 + 0.0003\xi_6 + 0.0471\xi_7 \\ &= (-0.3230; 1.8114; 2.3356). \end{aligned}$$

An investor who want to use portfolios in the set of the best portfolios based on second order dominance should share its capital as described in the above two portfolios. We can see that the most of the capital should be invested on asset 1; others assets share a few part of the capital. With $\xi^{(3)}$ and $\xi^{(4)}$ above, we expect to get values as described respectively on Table 2.

2.3.2 Comparisons of parameters of best portfolios with those of previous results

Our First Approach reaches to portfolios $\xi^{(1)}$ and $\xi^{(2)}$. We can see its effectiveness from Table 2. Our model improve the maximum expected value and the maximum skewness found by the previous

Table 2: Comparison with existed results

	Mean	Variance	Skewness	Kurtosis
Huang's model	1.60	0.7235	-0.7543	1.7972
Li et al's model	1.60	0.7019	-0.6823	1.7291
Fono et al's model	1.60	0.7018	-0.6823	1.7290
Our first portfolio, $\xi^{(1)}$	1.6074	0.7223	-0.7313	1.8175
Our second portfolio, $\xi^{(2)}$	1.6025	0.7147	-0.7187	1.7800
Our third portfolio, $\xi^{(3)}$	1.4000	0.4434	-0.3380	0.6922
Our fourth portfolio, $\xi^{(4)}$	1.4088	0.4604	-0.3557	0.7471

authors. The model improves again the kurtosis, and the variance found by Huang (2008).

Our Second Approach reaches to portfolios $\xi^{(3)}$ and $\xi^{(4)}$. Its effectiveness is shown from Table 2. The model improves the variance, the skewness and the kurtosis found by the previous authors. However the expected value is small comparing by one found by the previous authors. The model is very interesting for an investor who is fearing risk, because it improves very significantly the variance; which is the risk measure.

In the last Section, we rewrite in a simple way characterizations of the second order dominance for trapezoidal and triangular fuzzy variables proposed by Tassak et al. (2016), recalled in Theorem (1) in this paper.

3 Characterizations of Second Order Dominance for Trapezoidal and Triangular Fuzzy Variables

Proposition 1 (Characterization of the Second Order Dominance for Trapezoidal Fuzzy Variables) *Let $\xi_1 = (a_1, b_1, c_1, d_1)$ and $\xi_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, with Φ_1 and Φ_2 their corresponding credibility distribution functions, respectively. We have the following:*

1. *Given $(a_1 < b_1 < c_1 < d_1 \leq a_2 < b_2 < c_2 < d_2)$ or $(a_1 < b_1 < c_1 \leq a_2 < d_1 \leq b_2 < c_2 < d_2)$; then $\xi_1 \succ_2 \xi_2$ if and only if $\forall r \in \mathbb{R}, \Phi_2(r) \geq \Phi_1(r)$. This first case is related to non (zero) crossing point.*
2. *If $a_2 \leq a_1, b_2 \leq b_1, c_2 < c_1$ and $d_2 < d_1$ then $\xi_1 \succ_2 \xi_2$; with c_2 the only crossing point.*
3. *If $a_1 \leq a_2, b_1 \leq b_2, c_1 < c_2$ and $d_1 < d_2$ then $\xi_2 \succ_2 \xi_1$; with c_1 the only crossing point.*
4. *Given $a_2 < a_1 < b_1 < c_1 < d_1 \leq b_2 < c_2 < d_2$; then $\xi_1 \succ_2 \xi_2$ if and only if*
$$\int_{a_2}^{r_0} [\Phi_2(r) - \Phi_1(r)]dr \geq \int_{r_0}^{d_2} [\Phi_1(r) - \Phi_2(r)]dr,$$
with $r_0 \in [a_1, b_1]$ the only crossing point.
5. *Given $a_1 < a_2 < b_2 < b_1, c_1 = c_2$ and $d_1 = d_2$; then $\xi_2 \succ_2 \xi_1$ if and only if*
$$\int_{a_1}^{r_0} [\Phi_1(r) - \Phi_2(r)]dr \geq \int_{r_0}^{b_1} [\Phi_2(r) - \Phi_1(r)]dr,$$
with $r_0 \in [a_2, b_2]$, the only crossing point. Case (2.) up to case (5.) are related to one crossing point.
6. *Given $a_1 = a_2, b_1 = b_2, c_1 < c_2$ and $d_2 < d_1$; then $\xi_2 \succ_2 \xi_1$ if and only if*
$$\int_{c_1}^{r_0} [\Phi_1(r) - \Phi_2(r)]dr \geq \int_{r_0}^{d_1} [\Phi_2(r) - \Phi_1(r)]dr,$$
with c_1, and $r_0 \in [c_2, d_2]$ the crossing points. This case is related to two crossing points.

7. *Given* $a_1 < a_2, b_2 < b_1, c_1 < c_2$ *and* $d_2 < d_1$; *then* $\xi_2 \succ_2 \xi_1$ *if and only if*

   (a) $\int_{a_1}^{r_{01}} [\Phi_1(r) - \Phi_2(r)]dr > \int_{r_{01}}^{b_1} [\Phi_2(r) - \Phi_1(r)]dr,$

   (b) $\int_{a_1}^{r_{01}} [\Phi_1(r) - \Phi_2(r)]dr + \int_{c_1}^{r_{02}} [\Phi_1(r) - \Phi_2(r)]dr \geq \int_{r_{01}}^{b_1} [\Phi_2(r) - \Phi_1(r)]dr + \int_{r_{02}}^{d_1} [\Phi_2(r) - \Phi_1(r)]dr;$

   with c_1, *and* $r_{01} \in [a_2, b_2]$ *and* $r_{02} \in [c_2, d_2]$, *the crossing points.*

Remarks 1

For the case where there is no crossing point, case (1.), the second order dominance is equivalent to the first order dominance. Cases (2.) and (3.) are equivalent to the characterization of the first order dominance. For the first order dominance's definition and its characterization, we refer to Tassak et al. (2016).

Proof 1 i. *Given* $(a_1 < b_1 < c_1 < d_1 \leq a_2 < b_2 < c_2 < d_2)$ *or* $(a_1 < b_1 < c_1 \leq a_2 < b_2 < c_2 < d_2)$. *Assume that* $\xi_1 \succ_2 \xi_2$. *From Definition* (13), *we have*

$$\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0$$
$$\Phi_2(r) - \Phi_1(r) \geq 0$$
$$\Phi_2(r) \geq \Phi_1(r)\,,\ \forall r \in \mathbb{R} a.e.$$

So the necessary condition is proven by the definition of the second order dominance, Definition (13). *Conversely assume that* $\forall r \in \mathbb{R}$, $\Phi_2(r) \geq \Phi_1(r)$. *Which can be written as*

$\Phi_2(r) - \Phi_1(r) \geq 0$, $\forall r \in \mathbb{R}$. *By taking the integral from* $-\infty$ *up to* t, *with* $t \in \mathbb{R}$, *we get* $\int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0$. *Then by Definition* (13), *we conclude that* $\xi_1 \succ_2 \xi_2$. *The sufficient condition is also proven by the definition of the second order dominance, Definition* (13).

ii. *Suppose that* $a_2 \leq a_1, b_2 \leq b_1, c_2 < c_1$ *and* $d_2 < d_1$. *From Theorem* (1), *we have:*

- *The first assertion of the mentioned theorem is satisfy, because* $\exists c_2 \in \mathbb{R}$ *a crossing point such that* $\int_{-\infty}^{c_2} [\Phi_2(r) - \Phi_1(r)]dr \geq 0$.
- *The second assertion of the mentioned theorem is also true, that is* $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr \geq 0$. *This assertion can be written as*

$$\begin{aligned}
\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr &= \int_{-\infty}^{c_2} [\Phi_2(r) - \Phi_1(r)]dr + \int_{c_2}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr \\
&= \int_{-\infty}^{a_2} [\Phi_2(r) - \Phi_1(r)]dr + \int_{a_2}^{c_2} [\Phi_2(r) - \Phi_1(r)]dr \\
&+ \int_{c_2}^{d_1} [\Phi_2(r) - \Phi_1(r)]dr + \int_{d_1}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr \\
&= \int_{a_2}^{b_1} [\Phi_2(r) - \Phi_1(r)]dr + \int_{c_2}^{d_1} [\Phi_2(r) - \Phi_1(r)]dr > 0
\end{aligned}$$

The other cases can be proved in the same manner. □

Regards to Proposition (1); in the proposition below, we consider two triangular fuzzy numbers.

Proposition 2 (Characterization of the Second Order Dominance for Triangular Fuzzy Variables) *Let* $\xi_1 = (a_1, b_1, c_1)$ *and* $\xi_2 = (a_2, b_2, c_2)$ *be two triangular fuzzy numbers, with* Φ_1 *and* Φ_2 *their corresponding credibility distribution functions, respectively. We have the following:*

1. *If* $(a_2 < a_1 \leq b_2 < c_2 \leq b_1 < c_1)$ *or* $(a_2 < b_2 < c_2 \leq a_1 < b_1 < c_1)$ *or* $(a_2 < a_1 < b_1 = b_2 < c_2 < c_1)$, *then* $\xi_1 \succ_2 \xi_2$. *(no crossing point).*

2. *Given* $(a_2 < a_1 < b_1 = b_2 < c_1 < c_2)$ *or* $(a_2 < a_1 < b_1 < b_2 < c_1 < c_2)$, *then* $\xi_1 \succ_2 \xi_2$ *if and only if* $\int_{a_2}^{r_0}[\Phi_2(r) - \Phi_1(r)]dr \geq \int_{r_0}^{c_2}[\Phi_1(r) - \Phi_2(r)]dr$; *(with* $r_0 \in [a_1, b_1]$ *a crossing point which can be* b_1 *or strictly less that* b_1*).*
3. *Given* $(a_2 < a_1 < b_1 < b_2 < c_2 < c_1)$, *then* $\xi_1 \succ_2 \xi_2$ *if and only if* $\int_{a_2}^{r_{01}}[\Phi_2(r) - \Phi_1(r)]dr \geq \int_{r_{01}}^{r_{02}}[\Phi_1(r) - \Phi_2(r)]dr$, *(with* $r_{01} \in [a_1, b_1]$ *and* $r_{02} \in [b_2, c_2]$ *two crossing points).*

Case 1. is equivalent to the characterization of the first order dominance.

Proof 2 *The proof can be deduced from the proof of Proposition* (1).□

Regards to Proposition (2), we deduce this below corollary.

Corollary 1 *Let* $\xi_1 = (a_1, b_1, c_1)$ *and* $\xi_2 = (a_2, b_2, c_2)$ *be two symmetric triangular fuzzy numbers. Let* $supp(\xi_1)$ *and* $supp(\xi_2)$ *their corresponding supports, respectively. Assuming that the lengths of* $supp(\xi_1)$ *and* $supp(\xi_2)$ *are equals. We have then:*

1. *If* $(a_2 < a_1 \leq b_2 < c_2 \leq b_1 < c_1)$ *or* $(a_2 < b_2 < c_2 \leq a_1 < b_1 < c_1)$ *or* $(a_2 < a_1 < b_1 = b_2 < c_2 < c_1)$, *then* $\xi_1 \succ_2 \xi_2$.
2. *If* $(a_2 < a_1 < b_1 = b_2 < c_1 < c_2)$ *or* $(a_2 < a_1 < b_1 < b_2 < c_1 < c_2)$, *then* $\xi_1 \succ_2 \xi_2$.
3. *If* $(a_2 < a_1 < b_1 < b_2 < c_2 < c_1)$, *then* $\xi_1 \succ_2 \xi_2$.

4 Concluding remarks

In this paper, we have proposed two multiobjective problems (models) for portfolio selection with fuzzy return. The first model, called the first approach, is a multiobjective problem based on parameters of a fuzzy variable. This model is based on minimization of the first two even moments and on maximization of the first two odd moments of a fuzzy variable. It is not based on target values contrary to the previous models on parameters proposed by Huang (2008), Li et al. (2010) and Sadefo et al. (2012a). The second model, called second approach, is also a multiobjective problem based on the second order dominance of fuzzy variables.

Two numerical examples were solved using our proposed models. The first approach shows more effectiveness with improved expected value, variance and skewness, as compared to previous research by Huang (2008), Li et al. (2010) and Sadefo et al. (2012a). Even our second approach shows more effectiveness by improving the variance, the skewness and the kurtosis found by the previous research. Our second approach is very interesting for an investor who is fearing risk.

In this regards, one of the open question is: How can we do a comparison of fuzzy variables by considering the high order dominance? For instance, third order dominance or fourth order dominance, which leads the finding an application such as set of the best portfolios and an optimization model or a multiobjective model for choosing "good" portfolio.

References

B. Bouchon-Meunier. *La Logique Floue*. Presse Universitaires de France, quatrième edition, 1993.

C. Carlsson, R. Fullér, and P. Majlender. A possibilistic approach to selecting portfolios with highest utility score. *Fuzzy sets and systems, ELSEVIER*, 2001.

D. Dubois and H. Prade. *Fuzzy Sets and Systems, Theory and Applications*. Academic Press, New-York, Tokyo, Toronto, Sydney, San-Fransisco, 1980.

P. Gupta, M. K. Mehlawat, M. Inuiguchi, and S. Chandra. *Fuzzy Portfolio Optimization: Advances in Hybrid Multi-criteria Methodologies*. Springer-Verlag Berlin Heidelberg, 2014.

X. Huang. Mean-semivariance models for fuzzy portfolio selection. *Journal of Computational and Applied Mathematics, ELSEVIER*, 2008.

X. Huang. *Portfolio Analysis: From Probabilistic to Credibilistic and Uncertain Approaches*. Springer-Verlag Berlin Heidelberg, 2010.

M. Kanyinda, M. M. Rostin, and U. E. Lukata. Sur la multiplication des nombres flous detype l-r. *Annales de la Faculté des Sciences, Université de Kinshasa*, 1:33–40, 2014.

X. Li, Z. Qin, and S. Kar. Mean-variance-skewness model for portfolio selection with fuzzy returns. *European journal of Operational Research, ELSEVIER*, 2010.

B. Liu. *Uncertainty Theory*. Springer-Verlag Berlin, 2002.

B. Liu. *Theory and Practice of Uncertain Programming*. Springer-Verlag Berlin Heidelberg, 2009.

B. Liu and Y.-K. Liu. Expected value of fuzzy variable and fuzzy expected value models. *IEE Transactions on Fuzzy Systems 10, 445-450*, 2002.

H. Markowitz. Porfolio selection. *The journal of Finance*, 7:77–91, 1952.

H. Marling and S. Emanuelsson. The markowitz portfolio theory. 2012.

J. Sadefo, C. T. Deffo, and L. A. Fono. Moments and semi-moments for fuzzy portfolio selection. *Insurance: Mathematics and Economics, ELSEVIER*, 2012a.

J. Sadefo, C. D. Tassak, and L. A. Fono. Dominances on fuzzy variables based on credibility measure. *hal-00796215, HAL*, 2012b.

C. D. Tassak. Decision criteria on fuzzy variables for portfolio selection with fuzzy returns. *Ph.D. Thesis preparation, University of Yaoundé I-Cameroon*, 2016.

C. D. Tassak, J. Sadefo, L. A. Fono, and N. G. Andjiga. On characterization of order dominances on fuzzy variables for portfolio selection with fuzzy returns. *University of Douala-Cameroon, University of Yaoundé I-Cameroon and University of Guyanne-France*, 2016.

L. A. Zadeh. Fuzzy sets. *University of California, Berkeley, California*, 1965.