Holomorphic Functions of Bounded Type

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Abstract: We prove that if U is a balanced $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ - domain of holomorphy in T sirelson's space then the spectrum of $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ is identified with U. We show that if A is a bounded subset of a Banach space E, then $\widehat{A}_{\mathcal{H}_{(\alpha+\varepsilon)}(E)} = \widehat{A}_{\mathcal{P}(E)}$. Also we show theorems of Banach-Stone type for the algebras $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ and $\mathcal{H}_{(\alpha+\varepsilon)}(V)$.

Keywords: convex open subsets, Banach stone, holomorphic mappings.

Introduction

Let E be a Banach space and let U be an open subset of E. In [2, 15], it is proved that if E is Tsirelson's space, then the spectrum of $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ is identified with U, when $U = E$. In [11], J. Mujica generalized this result for absolutely convex open subsets of Tsirelson's space, and asks if the result can be improved for a more general class of open subsets of E, for instance, polynomially convex open subsets. In this Paper we give a partial answer to this question, i.e., we show that the result remains true for balanced $\mathcal{H}_{(\alpha+\varepsilon)}(U)$-domains of holomorphy on Tsirelson's space. we define $\mathcal{H}_{(\alpha+\varepsilon)}(U)$-convex open subsets and present properties and examples of such sets. We also give some auxiliary results. Most of them are generalizations to U-bounded sets of known results for compact sets. Also we show a corollary on finitely generated ideals of the algebra $\mathcal{H}_{(\alpha+\varepsilon)}(U)$. Finally we show algebras of holomorphic germs $\mathcal{H}(K)$ and $\mathcal{H}(L)$, improving results from [14].

Blanced Open Subspace and Continuous Mappings

We refer to [7,15] and [10] for background information on infinite
dimensional complex analysis. E and Fwill always denote Banach spaces. Let $\mathcal{P}(E;F)$ denote the Banach space of all continuous polynomials from E into F. $\mathcal{P}(^mE;F)$ denotes the Banach space of all continuous m-homogeneous polynomials from E into F.

$\mathcal{P}_f(^mE;F)$ denotes the subspace of $\mathcal{P}(^mE;F)$ generated by all polynomials of the form $P(x) = \varphi(x)^m b$, for all $x \in E$, where $\phi \in E'$ and $b \in F$.

Such polynomials are called of finite type. When $F = \mathbb{C}$, we write $\mathcal{P}(E)$, $\mathcal{P}(^mE)$ and $\mathcal{P}_f(^mE)$ instead of $\mathcal{P}(E;\mathbb{C})$, $\mathcal{P}(^mE;\mathbb{C})$ and $\mathcal{P}_f(^mE;\mathbb{C})$ respectively.

Let U be an open subset of E. We say that a subset $A \subset U$ is U-bouded if A is bounded and there exists $\varepsilon > 0$ such that $A + B(0,\varepsilon) \subset U$.

We will denote by $\mathcal{H}_{(\alpha+\varepsilon)}(U;F)$ the vector space of all holomorphic mappings $f : U \to F$ which are bounded on every U-bounded subset. Such mappings are called holomorphic mappings of bounded type. If $F = \mathbb{C}$, we write $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ instead of $\mathcal{H}_{(\alpha+\varepsilon)}(U;\mathbb{C})$. We denote by $\tau_{(\alpha+\varepsilon)}$ the topology on $\mathcal{H}_{(\alpha+\varepsilon)}(U;F)$ of the uniform convergence on all U-bounded subsets. $\mathcal{H}_{(\alpha+\varepsilon)}(U;F)$ is a Fréchet space for this topology, and like wise $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ is aFréchet algebra. If U is balanced, it follows from the Cauchy inequalities that the Taylor series of each $f \in \mathcal{H}_{(\alpha+\varepsilon)}(U;F)$ at the origin converges uniformly on each U- bounded subset. In particular, if ρ_U denotes the restriction of mappings to U, then $\rho_U(\mathcal{P}(E;F))$ is $\tau_{(\alpha+\varepsilon)}$-dense in $\mathcal{H}_{(\alpha+\varepsilon)}(U;F)$.

We denote by $S_{(\alpha+\varepsilon)}(U)$ the spectrum of the algebra $\mathcal{H}_{(\alpha+\varepsilon)}(U)$, i.e., the set of all continuous complex homomorphisms (and by that we mean linear and multiplicative) of $\mathcal{H}_{(\alpha+\varepsilon)}(U)$.

Every point of U can be associated with an element of $S_{(\alpha+\varepsilon)}(U)$ as follows: for each $z \in U$ fixed, let $\delta_z : \mathcal{H}_{(\alpha+\varepsilon)}(U) \to \mathbb{C}$ be defined by $\delta_z(f) = f(z)$, for all $f \in \mathcal{H}_{(\alpha+\varepsilon)}(U)$. Each δ_z is called evaluation at z. It is clear that $\delta_z \in S_{(\alpha+\varepsilon)}(U)$, for all $z \in U$, and the mapping $\delta : U \to S_{(\alpha+\varepsilon)}(U)$ is used in order to identify U

with the subset $\delta(U)$ of $S_{(a+\varepsilon)}(U)$. Note that δ is injective because the continuous linear forms already separate the points of E.

We will show that under certain hypotheses on E and U, all the elements of $S_{(a+\varepsilon)}(U)$ are evaluations at some point of U, and in this sense we say that $S_{(a+\varepsilon)}(U)$ is identified with $\delta(U)$.

In the following we give some needed results.

Let X be a subset of E, A be a subset of X, and $\mathcal{F} \subset \mathcal{C}(X)$. Then the $\mathcal{F}$-hull of A is the following set:

$$\widehat{A}_{\mathcal{F}} = \left\{x \in X: |f(x)| \leq \sup_{A}|f|, \text{for all } f \in \mathcal{F}\right\}.$$

Definition(1):

Let E be a Banach space and let U be an open subset of E. We say that U is:

(a) $\mathcal{P}_{(a+\varepsilon)}(E)$-convex if $\widehat{A}_{\mathcal{P}(E)} \cap U$ is U-bounded, for every U- bounded subset A;
(b) strongly $\mathcal{P}_{(a+\varepsilon)}(E)$-convex if $\widehat{A}_{\mathcal{P}(E)} \subset U$ and is U-bounded, for every U-bounded subset A;
(c) $\mathcal{H}_{(a+\varepsilon)}(E)$-convex if $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \cap U$ is U-bounded, for every U-bounded subset A;
(d) strongly $\mathcal{H}_{(a+\varepsilon)}(E)$-convex if $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ and is U-bounded, for every U-bounded subset A;
(e) $\mathcal{H}_{(a+\varepsilon)}(U)$-convex if $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(U)} \cap U$ is U-bounded, for every U-bounded subset A;

The following lemma shows that the notions of (strongly) $\mathcal{P}_{(a+\varepsilon)}(E)$-convex and (strongly) $\mathcal{H}_{(a+\varepsilon)}(E)$-convex set coincide.

Lemma (2):

Let A be a bounded subset of E. Then $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} = \widehat{A}_{\mathcal{P}(E)}$.

Proof:
Since $\mathcal{P}(E) \subset \mathcal{H}_{(a+\varepsilon)}(E)$, we have that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset \widehat{A}_{\mathcal{P}(E)}$. Now let us suppose that there exists $a \in \widehat{A}_{\mathcal{P}(E)}$ such that $a \notin \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$. Let $f \in \mathcal{H}_{(a+\varepsilon)}(E)$ be such that $|f(a)| > \sup_A|f|$. since $\widehat{A}_{\mathcal{P}(E)}$ is bounded and $\mathcal{P}(E)$ is dense in $\mathcal{H}_{(a+\varepsilon)}(E)$ for the $\tau_{(a+\varepsilon)}$ topology. Given $\varepsilon > 0$ there exists $P \in \mathcal{P}(E)$ such that $\sup_{\widehat{A}_{\mathcal{P}(E)}}|f-P| < \frac{\varepsilon}{2}$. In particular we have that
$\sup_A|p| \leq \sup_A|p-f| + \sup_A|f| < \frac{\varepsilon}{2} + \sup_A|f|$.
Finally we get that $|f(a)| \leq |f(a) - p(a)| + |p(a)| < \frac{\varepsilon}{2} + \sup_A|P| < \varepsilon + \sup_A|f|$, for all $\varepsilon > 0$, which is a contradiction.

Lemma (3):
If $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$, for every U-bounded subset A, then U is strongly $\mathcal{H}_{(a+\varepsilon)}(E)$-cnvex.

Proof:
We follow ideas of [9]. Let A be a U-bounded subset. We must show that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ is U-bounded. Since it is clear that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ is bounded, it remains to show that there exists $\varepsilon > 0$ such that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} + B(0,\varepsilon) \subset U$. Let $\varepsilon > 0$ be such that $A + B(0,\varepsilon)$ is U-bounded. Then $(A+B(0,\varepsilon))^{\wedge}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$. Let $y \in \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$, $t \in B(0,\varepsilon)$ and $0 < \theta < 1$. Then for each $f \in \mathcal{H}_{(a+\varepsilon)}(E)$ we have that

$$|f(y+\theta t)| \leq \sum_{m=0}^{\infty} \theta^m |p_t^m(f)(y)| \leq \sum_{m=0}^{\infty} \theta^m \sup_{A}|p_t^m(f)|$$

$$\leq (1-\theta)^{-1} \sup_{\Lambda+B(0,\varepsilon)} |(f)|,$$

where the second inequality follows because $P_t^m(f) \in \mathcal{H}_{(a+\varepsilon)}(E)$ and $y \in \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$. The third inequality follows by applying [10], with $t \in B(0,\varepsilon)$ and $r = 1$.
Next we apply the above inequality to f^n, take n -th roots and let $n \to \infty$ to get that $|f(y+\theta t)| \leq \sup_{A+B(0,\varepsilon)}|f|$, that is, $y + \theta t \in (A+B(0,\varepsilon))^{\wedge}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$. By letting $\theta \to 1$ we have that $y + t \in U$, and the conclusion follows.

Lemma (4):
Let $\mathcal{F} \in \mathcal{H}_{(a+\varepsilon)}(E)$ be a family with the property that the function $x \mapsto f(\lambda x)$ is an element of $\mathcal{F}$, for every $f \in \mathcal{F}$ and $|\lambda| \leq 1$. Let $A \subseteq E$ be a balanced subset. Then $\widehat{A}_{\mathcal{F}}$ is balanced.

Proof:
Let $f \in \mathcal{F}$. For each $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$, let $f_\lambda \in \mathcal{F}$ be such that $f_\lambda(x) = f(\lambda x)$, for all $x \in E$. Let $y \in \widehat{A}_{\mathcal{F}}$. Then $|f(\lambda y)| = |f_\lambda(y)| \leq \sup_A|f_\lambda| \leq \sup_A|f|$, proving that $\lambda y \in \widehat{A}_{\mathcal{F}}$, and hence $\widehat{A}_{\mathcal{F}}$ is balanced.

It is clear that strongly $\mathcal{H}_{(a+\varepsilon)}(E)$-convex open subsets are always $\mathcal{H}_{(a+\varepsilon)}(E)$-convex. Also, it is easy to see that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \cap U$, and hence we have that $\mathcal{H}_{(a+\varepsilon)}(E)$-convex open subsets are always $\mathcal{H}_{(a+\varepsilon)}(U)$-convex . The next Proposition shows that if U is balanced, then all these notions coincide.

Propositions (5):
Let $U \subset E$ be a balanced open subset. Then the following conditions
are equivalent.
(a) U is strongly $\mathcal{P}_{(a+\varepsilon)}(E)$-convex.
(b) U is stongly $\mathcal{H}_{(a+\varepsilon)}(E)$-convex.
(c) U is $\mathcal{P}_{(a+\varepsilon)}(E)$-convex.
(d) U is $\mathcal{H}_{(a+\varepsilon)}(E)$-convex.
(e) U is $\mathcal{H}_{(a+\varepsilon)}(U)$-convex.

Proof:
The implication (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d) were proved in Lemma (2). The implication (b) $\Rightarrow$ (d) $\Rightarrow$ (e) were commented above.
$(e) \Rightarrow (d)$ we show that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(U)} = \widehat{A}_{\mathcal{P}(E)} \cap U = \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \cap U$, for every U-bounded subset A, and then conclude that U is $\mathcal{H}_{(a+\varepsilon)}(E)$-convex. Let $y \in \widehat{A}_{\mathcal{P}(E)} \cap U$ and we show that $y \in \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(U)}$. Let $f \in \mathcal{H}_{(a+\varepsilon)}(E)$ fixed. Since U is $\mathcal{H}_{(a+\varepsilon)}(E)$-convex, the set $B = \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(U)} \cup \{y\}$ is U-bounded, and since U is balanced, given $\varepsilon > 0$, there is $P \in \mathcal{P}(E)$ such that

$$\sup_B |f - P| < \frac{\varepsilon}{2}.$$

Then

$$|f(y)| \leq |f(y) - P(y)| + |P(y)| < \frac{\varepsilon}{2} + \sup_A |P|.$$

On other hand:

$$\sup_A |P| \leq \sup_A |f - P| + \sup_A |f| < \frac{\varepsilon}{2} + \sup_A |f|.$$

And finally we get that $|f(y)| < \varepsilon + sup_A|f|$, for all $\varepsilon > 0$, which implies that $y \in \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(U)}$.
$(d) \Rightarrow (b)$ let A be an U-bounded subset. By Lemma(3), it suffices to prove that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$. First we assume that A is balanced. Let $x \in \widehat{A}_{\mathcal{P}(E)}$. Define $D_1 = \{\lambda \in \mathbb{C}: |\lambda| \leq 1\}$ and $D = \{\lambda \in D_1: \lambda x \in U\}$. Then D is a disk centered at the origin because U is balanced, and D is an open subset of D_1 because U is open. Let $\varepsilon > 0$ be such that $\widehat{A}_{\mathcal{P}(E)} \cap U + B(0,\varepsilon) \subset U$. Let $\lambda \in D_1, \lambda x \in U$, and let $\mu \in D_1$ be such that $|\mu - \lambda| \|x\| < \varepsilon$.Then $\lambda x \in \widehat{A}_{\mathcal{P}(E)} \cap U$ because $x \in \widehat{A}_{\mathcal{P}(E)}$ and $\widehat{A}_{\mathcal{P}(E)}$is balanced by Lemma(4). Furthermore $\|\mu x - \lambda x\| < \varepsilon$, hence $\mu x \in \widehat{A}_{\mathcal{P}(E)} \cap U + B(0,\varepsilon)$, and therefore $\mu x \in U$. This implies that any point on the boundary of D belongs to D, and D is an open and closed subset of D_1, and therefore $D = D_1$. It follows that $x = 1x \in U$. Since this holds for any $x \in \widehat{A}_{\mathcal{P}(E)}$,we have proved that $\widehat{A}_{\mathcal{P}(E)} \subset U$.

If A is not balanced, we consider $B = ba(A)$, the balanced hull of A.

It follows by [5] that, B is a balanced U-bounded subset. Then we apply the arguments above and get that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset \widehat{B}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$.

Corollary (6):
Let A be a bounded subset of E. Then $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} = \widehat{A}_{\mathcal{P}(E)}$.

Proof:
Since $\mathcal{P}(E) \subset \mathcal{H}_{(a+\epsilon)}(E)$, we have that $\widehat{A}_{\mathcal{H}_{(a+\epsilon)}(E)} \subset \widehat{A}_{\mathcal{P}(E)}$. For $a \in \widehat{A}_{\mathcal{P}(E)}$ such that $a \notin \widehat{A}_{\mathcal{H}_{(a+\epsilon)}(E)}$, let $f_j \in \mathcal{H}_{(a+\epsilon)}(E)$ be such that $\sum_{j=1}^{n}|f_j(a)| > \sup_A \sum_{j=1}^{n}|f_j|$. Since $\widehat{A}_{\mathcal{P}(E)}$ is bounded and $\mathcal{P}(E)$ is dense in $\mathcal{H}_{(a+\epsilon)}(E)$ for the $\tau_{(a+\epsilon)}$ topology. Given $\epsilon > 0$ there exists $P^j \in \mathcal{P}(E)$ such that $\sup_{\widehat{A}_{\mathcal{P}(E)}} \left|\sum_{j=1}^{n} f^j - \sum_{j=1}^{n} P^j\right| < \frac{\epsilon}{2}$.

We get in particular,

$$\sup_A \sum_{j=1}^{n} |p^j| \leq \sup_A \sum_{j=1}^{n} |p^j - f_j| \leq \sup_A \sum_{j=1}^{n} |f_j|$$

$$< \frac{\epsilon}{2} + \sup_A \sum_{j=1}^{n} |f_j|.$$

Hence

$$\sum_{j=1}^{n} |f_j(a)| \leq \sum_{j=1}^{n} |f_j(a) - p^j(a)| + \sum_{j=1}^{n} |p^j(a)|$$

$$< \frac{\epsilon}{2} + \sup_A \sum_{j=1}^{n} |p^j|, \qquad for\ \epsilon > 0.$$

which is a contradiction.

Corollary (7):
Let $\mathcal{F} \in \mathcal{H}_{(a+\varepsilon)}(E)$ be a family with the property that the function $x \mapsto f((\lambda_1 + \lambda_2 + \cdots + \lambda_n)x)$ is an element of $\mathcal{F}$, for every $f \in \mathcal{F}$ and $|\lambda_1 + \lambda_2 + \cdots + \lambda_n| \leq 1$. Let $A \subseteq E$ be a balanced subset. Then $\widehat{A}_{\mathcal{F}}$ is balanced.

Proof:
Let $f \in \mathcal{F}$, for each $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ such that $|\lambda_1 + \cdots + \lambda_n| \leq 1$, let $f_{(\lambda_1 + \cdots + \lambda_n)} \in \mathcal{F}$ be such that

$f_{(\lambda_1+\ldots+\lambda_n)}(x) = f((\lambda_1 + \cdots + \lambda_n)x)$, for any $x \in E$. Let $y \in \widehat{A}_{\mathcal{F}}$. Then

$$|f((\lambda_1 + \lambda_2 + \cdots + \lambda_n)y)| = |f_{(\lambda_1+\ldots+\lambda_n)}(y)| \leq \sup_A |f_{(\lambda_1+\lambda_2+\cdots+\lambda_n)}| \leq \sup_A |f|,$$

proving that $(\lambda_1 + \cdots + \lambda_n)y \in \widehat{A}_{\mathcal{F}}$, and hence $\widehat{A}_{\mathcal{F}}$ is balanced.

Next we give some examples.

Example (8):

Let $P \in \mathcal{P}(^mE;F)$ and let $U = \{x \in E: \|P(x)\| < 1\}$. Then U is a balanced $\mathcal{H}_{(a+\varepsilon)}(E)$-convex open set.

Proof:

Clearly U is a balanced open set. Let A be an U-bounded subset of U. Let $\varepsilon > 0$ denote the distance from A to the boundary of U, and let $r = \sup_{x \in A}\|x\|$. If $x \in A$ and $1 \leq \lambda < 1 + \frac{\varepsilon}{r}$, then $\|\lambda x - x\| = |\lambda - 1|\|x\| < \varepsilon$, hence $\lambda x \in U$, and therefore

$$\|P(x)\| = \left\|P\left(\left(\tfrac{1}{\lambda}\right)\lambda x\right)\right\| = \lambda^{-m}\|P(\lambda x)\| < \lambda^{-m}.$$

Taking in the right-hand side the infimum over all λ such that $1 \leq \lambda < 1 + \frac{\varepsilon}{r}$, we conclude that $\|P(x)\| \leq c := (1 + \frac{\varepsilon}{r})^{-m} < 1$ for every $x \in A$.

Let us show that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$. Let $y \in \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ and $\varphi \in F'$. Then $\varphi \circ P \in \mathcal{H}_{(a+\varepsilon)}(E)$ and hence $|\varphi \circ P(y)| \leq \sup_A|\varphi \circ P|$.

Now

$$\|P(y)\| = \sup_{\varphi \in B_{F'}} |\varphi(P(y))| \leq \sup_{\varphi \in B_{F'},\, x \in A} \sup|\varphi(P(x))| \leq \sup\|P(x)\| = c < 1,$$

and hence $y \in U$.

This shows that $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ is U-bounded, because if $0 < c < 1$, then every bounded subset of $\{x \in E: \|P(x)\| \leq c\}$ is U-bounded. Hence U is strongly $\mathcal{H}_{(a+\varepsilon)}(E)$-convex by Lemma (3). Finally U is $\mathcal{H}_{(a+\varepsilon)}(U)$-convex by Proposition (5).

Corollary (9):

Let $P \in \mathcal{P}(^mE)$ and let $U = \{x \in E: |P(x)| < 1\}$. Then U is a balanced $\mathcal{H}_{(a+\varepsilon)}(U)$-convex open set.

Corollary (10):

Let $A \in \mathcal{L}(E_1, \ldots, E_m; F)$ and $E = E_1 \times \ldots \times E_m$. Then

$$U = \{(x_1, \ldots, x_m) \in E: \|A(x_1, \ldots, x_m)\| < 1\}$$

is a balanced $\mathcal{H}_{(a+\varepsilon)}(U)$-convex open set.

Corollary (11):

Let $U = \{(x, \lambda) \in E \times \mathbb{C}: \|\lambda x\| < 1\}$. Then U is a balanced $\mathcal{H}_{(a+\varepsilon)}(U)$-convex open set. In [13], B. Tsirelson constructed a reflexive Banach space X, with an unconditional Schauder basis, that does not contain any subspace which is isomorphic to c_0 or to any ℓ_p. R. A lencar, R. Aron and S. Dineen proved in [1] that $\mathcal{P}_f(^mX)$ is norm-dense in $\mathcal{P}(^mX)$, for all $m \in \mathbb{N}$. Inspired by this result, we will say that a Banach space E is a Tsirelson-like space if E is reflexive and $\mathcal{P}_f(^mE)$ is norm-dense in $\mathcal{P}(^mE)$, for all $m \in \mathbb{N}$.

We have The following theorem.

Theorem (12):

Let E be a Tsirelson-like space, and let U be a balanced $\mathcal{H}_{(a+\varepsilon)}(U)$ convex open subset of E. Then the spectrum of $\mathcal{H}_{(a+\varepsilon)}(U)$is identified with U.

Proof:

Since U is balanced and $\mathcal{H}_{(a+\varepsilon)}(U)$-convex, it follows by Proposition (5) that U is strongly $\mathcal{H}_{(a+\varepsilon)}(U)$-convex. Now we follow the ideas of [11]. Let $T: \mathcal{H}_{(a+\varepsilon)}(U) \to \mathbb{C}$ be a continuous homomorphism. Then there exists $C > 0$ and a U-bounded subset $A \subset U$ such that

$$|T(f)| \leq C \sup_A |f|, \quad \text{for all } f \in \mathcal{H}_{(a+\varepsilon)}(U).$$

Since T is multiplicative, we have that $|T(f)|^n = |T(f^n)| \leq C \sup_A |f|^n$ for every $n \in \mathbb{N}$.

Taking n -th roots and making $n \to \infty$ we conclude that actually $C = 1$. Let $r > 0$ such that $A \subset B(0,r)$. In particular, we have that $|T(f)| \leq \sup_A|f| \leq \sup_{B(0,r)}|f|$, for all $f \in E'$. Hence we have that $T|_{E'} \in E'' = E$, so there exists a unique $a \in E$ such that $T(f) = f(a)$, for all $f \in E'$, and hence $T(P) = P(a)$, for all $P \in \mathcal{P}_f(^mE)$, for all $m \in \mathbb{N}$. Since $\mathcal{P}_f(^mE)$ is norm-dense in $\mathcal{P}(^mE)$, for all $m \in \mathbb{N}$, it follows that $T(P) = P(a)$, for all $P \in \mathcal{P}(E)$. Then we have that $|P(a)| = |P(f)| \leq \sup_A|P|$, for all $P \in \mathcal{P}(E)$, which implies that $a \in \widehat{A}_{\mathcal{P}(E)} = \widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$. Since U balanced, we have that $\mathcal{P}(E)$ is $\tau_{(a+\varepsilon)}$-dense in $\mathcal{H}_{(a+\varepsilon)}(E)$, and then we conclude that $T(f) = f(a)$, for all $f \in \mathcal{H}_{(a+\varepsilon)}(E)$, proving the Theorem.

Definition (13):

Let E be a Banach space and let U be an open subset of E. We say that U is a $\mathcal{H}_{(a+\varepsilon)}(U)$-domain of holomorphy if there are no open sets V and W in E with the following properties:

(a) V is connected and not contained in U;
(b) $\emptyset \neq W \subset U \cap V$;
(c) for each $f \in \mathcal{H}_{(a+\varepsilon)}(U)$ there exists $\tilde{f} \in \mathcal{H}(V)$ such that $\tilde{f} = f$ on W.

The following corollary is the announced result for balanced $\mathcal{H}_{(a+\varepsilon)}(U)$-domains of holomorphy.

Corollary (14):
Let E be a Tsirlson-like space, and let U be a balanced $\mathcal{H}_{(a+\varepsilon)}(U)$-domain of holomorphy in E. Then the spectrum of $\mathcal{H}_{(a+\varepsilon)}(U)$ is identified with U.
The following result is a consequence of Corollary (14). It says that under the hypotheses of Corollary(14), every proper finitely generated ideal of $\mathcal{H}_{(a+\varepsilon)}(U)$ has a common zero.

Theorem (15):
Let E be a Tsilrson-like space. Let $U \subset E$ be a balanced $\mathcal{H}_{(a+\varepsilon)}(U)$-domain of holomorphy. Then given $f_1, \dots, f_n \in \mathcal{H}_{(a+\varepsilon)}(U)$ without common zeros, we can find $g_1, \dots, g_n \in \mathcal{H}_{(a+\varepsilon)}(U)$ such that
$\sum_{i=1}^{n} f_i g_i = 1$.

In [3], S. Banach proved that two compact metric spaces X and Y are homomorphic if and only if the Banach algebras $\mathcal{C}(X)$, $\mathcal{C}(Y)$ are isometrically isomorphic. M.H. Stone, in [12] , generized this result to arbitrary compact Hausdorff topological spaces, the well-known Banach –Stone theorem.

Algebras of holomorphic germs

In [14], D.M. Vieira presents similar results for algebras of holomorphic functions of bounded type, using results on the spectrum of such algebras. More specifically, let E and F be reflexive spaces, one of them a Tsirelson-like space, and let $U \subset E$ and $V \subset F$ be absolutely convex open subsets. Then it is shown that the algebras $\mathcal{H}_{(a+\varepsilon)}(U)$ and $\mathcal{H}_{(a+\varepsilon)}(V)$ are topolo-gically isomorphic, if and only if there is a special type of holomorphic mappings between U and V. To show these results we use the characteriz-ation of the spectra of $\mathcal{H}_{(a+\varepsilon)}(U)$ with U due to J. Mujica, [11].
Now we are going to generalize this result to balanced $\mathcal{H}_{(a+\varepsilon)}(U)$-domains of holomorphy, using the characterization of the spectrum of $\mathcal{H}_{(a+\varepsilon)}(U)$.

Let E and F be Banach spaces, and $U \subset E$ and $V \subset F$ be open subsets of E and F, respectively. We denote by $\mathcal{H}_{(a+\varepsilon)}(V,U)$ the set of all holomorphic mappings $\varphi : V \to E$, with $\varphi(V) \subset U$, such that φ maps V-bounded subsets into U-bounded subsets.

Theorem (16):
Let E and F be reflexive Banach spaces, one of them a Tsirelson-like space. Let $U \subset E$ and $V \subset F$ be balanced $\mathcal{H}_{(a+\varepsilon)}$-domains of holomorphy. Then the following conditions are equivalent.

(a) There exists a bijective mapping $\varphi : V \to U$ such that $\varphi \in \mathcal{H}_{(a+\varepsilon)}(V,U)$ and $\varphi^{-1} \in \mathcal{H}_{(a+\varepsilon)}(U,V)$.
(b) the algebras $\mathcal{H}_{(a+\varepsilon)}(U)$ and $\mathcal{H}_{(a+\varepsilon)}(V)$ are topologically isomorphic.

In [14] it is shown that if $K \subset E$ and $L \subset F$ are absolutely convex compact subsets of Tsirelson-like spaces, then the algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic if and only if K and L are biholomorphically equivalent.

The key to the proof of such result is a theorem of Banach-Stone type between algebras of holomorphic functions of bounded type [14]. We are going to present a generalization of this result to greater class of compact sets, using Theorem (15).

Let E be a Banach space, and let $K \subset E$ be a compact subset. We define $\mathcal{H}(K)$ to be the algebra of all functions that are holomorphic on some open neighborhood of K. The elements of $\mathcal{H}(K)$ are called germs of holomorphic functions. We endow $\mathcal{H}(K)$ with the locally convex inductive limit of the locally convex algebras$(\mathcal{H}(U), \tau_\omega)$, where U varies among the open subsets of E such that $K \subset U$. If $U_n = K + B(0,\frac{1}{n})$, for all $n \in \mathbb{N}$, then it is easy to see that:

$$(\mathcal{H}(K), \tau_\omega) = \varinjlim_{n\in\mathbb{N}} \mathcal{H}_{(a+\varepsilon)}(U_n).$$

Definition (17):

Let E be a Banach space, let K be a compact subset of E and let $m \in \mathbb{N}$. We say that K is $\mathcal{P}(^m E)$-convex if $K = \widehat{K}_{\mathcal{P}(^m E)}$.

Before we present examples of balanced $\mathcal{P}(^m E)$-convex compact sets, we shall need the next lemma. If A is a subset of a Banach space, we denote by $\overline{\Gamma}(A)$ the closed, absolutely convex hull of A.

Lemma(18):
Let E be a Banach space and let A be a bounded subset of E. Then
$\widehat{A}_{\mathcal{P}_f(^m E)} \subset \overline{\Gamma}(A)$, for all $m \in \mathbb{N}$.
Proof:
Let $y \notin \overline{\Gamma}(A)$. By the Hahn-Banach Theorem, there exists $\varphi \in E'$ such that $|\varphi(y)| > \sup_{x\in\overline{\Gamma}(A)} |\varphi(x)|$.
Hence $|\varphi^m(y)| > \sup_{x\in\overline{\Gamma}(A)} |\varphi^m(x)| \geq \sup_A |\varphi^m|$, which shows that $y \notin \widehat{A}_{\mathcal{P}_f(^m E)}$.

Example (19):
Every absolutely convex compact subset of Banach space E is $\mathcal{P}(^m E)$-convex, for all $m \in \mathbb{N}$.

Proof:
Let $K \subset E$ be an absolutely convex compact set. Since $\mathcal{P}_f(^mE) \subset \mathcal{P}(^mE)$, we have that $\widehat{K}_{\mathcal{P}(^mE)} \subset \widehat{K}_{\mathcal{P}_f(^mE)} \subset \overline{\Gamma}(K) = K$, where the last inclusion follows by Lemma(18).

Example (20):
Let E be a Banach space, and $L \subset E$ be a compact, balanced and $\mathcal{P}(^mE)$-convex set. Let $P \in \mathcal{P}(^mE)$. Then $K = \{\, x \in L\colon |P(x)| \le 1\}$ is compact, balanced and $\mathcal{P}(^mE)$-convex.

Theorem (21):
Let E be a Banach space and let K be a compact, balanced and $\mathcal{P}(^mE)$-convex subset of E, for some $m \in \mathbb{N}$. Let U be an open subset of E such that $K \subset U$. Then there exists an open set $V \subset E$ which is balanced and $\mathcal{H}_{(a+\varepsilon)}(V)$-convex, such that $K \subset V \subset U$.

Proof:
We begin with a slight modification of [10]. If $\overline{\Gamma}(K) \subset U$, then we take $V = \overline{\Gamma}(K) + B(0,\varepsilon)$, where ε is such that $\overline{\Gamma}(K) + B(0,\varepsilon) \subset U$. If $\overline{\Gamma}(K)$ is not contained in U, then for each $a \in \overline{\Gamma}(K) \setminus U$ there is $P \in \mathcal{P}(^mE)$ such that $\sup_K |P| < 1 < |P(a)|$. Since $\overline{\Gamma}(K) \setminus U$ is compact, we can find polynomials $P_1, \ldots, P_k \in \mathcal{P}(^mE)$ such that

$$\overline{\Gamma}(K) \setminus U \subset \bigcup_{j=1}^{k} \{x \in E\colon |P_j(x)| > 1\}.$$

Now it is easy to see that $\{x \in \overline{\Gamma}(K)\colon |P_j(x)| \le 1 \text{, for } j = 1, \ldots, k\} \subset U$.
Next we follow the arguments of [10], finding a positive number $\delta > 0$ such that:

$$V = (\overline{\Gamma}(K) + B(0,\varepsilon)) \cap \{x \in E\colon |P_j(x)| < 1 \text{, for } j = 1, \ldots, k\} \subset U$$
.
Now V is balanced and $\mathcal{H}_{(a+\varepsilon)}(V)$-convex, by Corollary (9).

Let E and F be Banach spaces, and let $K \subset E$ and $L \subset F$ be compact subsets. We say that K and L are biholomorphically equivalent if there exist open subsets $U \subset E$ and $V \subset F$ with $K \subset U$ and $L \subset V$ and a biholomorphic mapping $\varphi\colon V \to U$ such that $\varphi(L) = K$. The next theorem is the announced result for algebras of holomorphic germs, and generalizes [14].

Theorem (22):
Let E and F be Tsirelson-like spaces. Let $K \subset E$ and $L \subset F$ be balanced compact subsets, such that K is $\mathcal{P}(^mE)$-convex, and L is $\mathcal{P}(^kF)$-convex, for some $m, k \in \mathbb{N}$. Then the following conditions are equivalent.

(a) K and L are biholomorphically equivalent.
(b) The algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic.

Proof:
$(a) \Rightarrow (b)$[14] applies.
$(b) \Rightarrow (a)$ We claim tat $\mathcal{H}(K)$ is the inductive limit of a sequence of Fréchet spaces $\mathcal{H}_{(a+\varepsilon)}(V_n)$, where each V_n is balanced and $\mathcal{P}_b(E)$-convex (and the same for $\mathcal{H}(L)$). Indeed, let $U_n = K + B(0;\frac{1}{n})$, for every $n \in \mathbb{N}$. Appling Theorem (21), for each $n \in \mathbb{N}$ there exists a balanced $\mathcal{H}_b$-convex open subset V_n such that $K \subset V_n \subset U_n$. Since $\mathcal{H}(K) = \lim_{\overrightarrow{n \in \mathbb{N}}} \mathcal{H}_b(U_n)$, and the inclusion $\mathcal{H}_{(a+\varepsilon)}(U_n) \hookrightarrow \mathcal{H}_{(a+\varepsilon)}(V_n)$ is continuous, we have that $\mathcal{H}(K) = \lim_{\overrightarrow{n \in \mathbb{N}}} \mathcal{H}_b(V_n)$, and our claim is proved. Next we apply the same arguments of $(b) \Rightarrow (a)$ of [14].

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