

# Holomorphic Functions of Bounded Type

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**Abstract:** We prove that if  $U$  is a balanced  $\mathcal{H}_{(a+\varepsilon)}(U)$ -domain of holomorphy in Tsirelson's space then the spectrum of  $\mathcal{H}_{(a+\varepsilon)}(U)$  is identified with  $U$ . We show that if  $A$  is a bounded subset of a Banach space  $E$ , then  $\widehat{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} = \widehat{A}_{\mathcal{P}(E)}$ . Also we show theorems of Banach-Stone type for the algebras  $\mathcal{H}_{(a+\varepsilon)}(U)$  and  $\mathcal{H}_{(a+\varepsilon)}(V)$ .

**Keywords:** convex open subsets, Banach stone, holomorphic mappings.

## Introduction

Let  $E$  be a Banach space and let  $U$  be an open subset of  $E$ . In [2, 15], it is proved that if  $E$  is Tsirelson's space, then the spectrum of  $\mathcal{H}_{(a+\varepsilon)}(U)$  is identified with  $U$ , when  $U = E$ . In [11], J. Mujica generalized this result for absolutely convex open subsets of Tsirelson's space, and asks if the result can be improved for a more general class of open subsets of  $E$ , for instance, polynomially convex open subsets. In this Paper we give a partial answer to this question, i.e., we show that the result remains true for balanced  $\mathcal{H}_{(a+\varepsilon)}(U)$ -domains of holomorphy on Tsirelson's space. we define  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex open subsets and present properties and examples of such sets. We also give some auxiliary results. Most of them are generalizations to  $U$ -bounded sets of known results for compact sets. Also we show a corollary on finitely generated ideals of the algebra  $\mathcal{H}_{(a+\varepsilon)}(U)$ . Finally we show algebras of holomorphic germs  $\mathcal{H}(K)$  and  $\mathcal{H}(L)$ , improving results from [14].

## Blanced Open Subspace and Continuous Mappings

We refer to [7,15] and [10] for background information on infinite dimensional complex analysis.  $E$  and  $F$  will always denote Banach spaces. Let  $\mathcal{P}(E; F)$  denote the Banach space of all continuous polynomials from  $E$  into  $F$ .  $\mathcal{P}^m(E; F)$  denotes the Banach space of all continuous  $m$ -homogeneous polynomials from  $E$  into  $F$ .

$\mathcal{P}_f^m(E; F)$  denotes the subspace of  $\mathcal{P}^m(E; F)$  generated by all polynomials of the form  $P(x) = \phi(x)^m b$ , for all  $x \in E$ , where  $\phi \in E'$  and  $b \in F$ .

Such polynomials are called of finite type. When  $F = \mathbb{C}$ , we write  $\mathcal{P}(E)$ ,  $\mathcal{P}^m(E)$  and  $\mathcal{P}_f^m(E)$  instead of  $\mathcal{P}(E; \mathbb{C})$ ,  $\mathcal{P}^m(E; \mathbb{C})$  and  $\mathcal{P}_f^m(E; \mathbb{C})$  respectively.

Let  $U$  be an open subset of  $E$ . We say that a subset  $A \subset U$  is  $U$ -bounded if  $A$  is bounded and there exists  $\varepsilon > 0$  such that  $A + B(0, \varepsilon) \subset U$ .

We will denote by  $\mathcal{H}_{(a+\varepsilon)}(U; F)$  the vector space of all holomorphic mappings  $f : U \rightarrow F$  which are bounded on every  $U$ -bounded subset. Such mappings are called holomorphic mappings of bounded type. If  $F = \mathbb{C}$ , we write  $\mathcal{H}_{(a+\varepsilon)}(U)$  instead of  $\mathcal{H}_{(a+\varepsilon)}(U; \mathbb{C})$ . We denote by  $\tau_{(a+\varepsilon)}$  the topology on  $\mathcal{H}_{(a+\varepsilon)}(U; F)$  of the uniform convergence on all  $U$ -bounded subsets.  $\mathcal{H}_{(a+\varepsilon)}(U; F)$  is a Fréchet space for this topology, and like wise  $\mathcal{H}_{(a+\varepsilon)}(U)$  is a Fréchet algebra. If  $U$  is balanced, it follows from the Cauchy inequalities that the Taylor series of each  $f \in \mathcal{H}_{(a+\varepsilon)}(U; F)$  at the origin converges uniformly on each  $U$ -bounded subset. In particular, if  $\rho_U$  denotes the restriction of mappings to  $U$ , then  $\rho_U(\mathcal{P}(E; F))$  is  $\tau_{(a+\varepsilon)}$ -dense in  $\mathcal{H}_{(a+\varepsilon)}(U; F)$ .

We denote by  $S_{(a+\varepsilon)}(U)$  the spectrum of the algebra  $\mathcal{H}_{(a+\varepsilon)}(U)$ , i.e., the set of all continuous complex homomorphisms (and by that we mean linear and multiplicative) of  $\mathcal{H}_{(a+\varepsilon)}(U)$ .

Every point of  $U$  can be associated with an element of  $S_{(a+\varepsilon)}(U)$  as follows: for each  $z \in U$  fixed, let  $\delta_z : \mathcal{H}_{(a+\varepsilon)}(U) \rightarrow \mathbb{C}$  be defined by  $\delta_z(f) = f(z)$ , for all  $f \in \mathcal{H}_{(a+\varepsilon)}(U)$ . Each  $\delta_z$  is called evaluation at  $z$ . It is clear that  $\delta_z \in S_{(a+\varepsilon)}(U)$ , for all  $z \in U$ , and the mapping  $\delta : U \rightarrow S_{(a+\varepsilon)}(U)$  is used in order to identify  $U$

with the subset  $\delta(U)$  of  $S_{(a+\varepsilon)}(U)$ . Note that  $\delta$  is injective because the continuous linear forms already separate the points of  $E$ .

We will show that under certain hypotheses on  $E$  and  $U$ , all the elements of  $S_{(a+\varepsilon)}(U)$  are evaluations at some point of  $U$ , and in this sense we say that  $S_{(a+\varepsilon)}(U)$  is identified with  $\delta(U)$ .

In the following we give some needed results.

Let  $X$  be a subset of  $E$ ,  $A$  be a subset of  $X$ , and  $\mathcal{F} \subset \mathcal{C}(X)$ . Then the  $\mathcal{F}$ -hull of  $A$  is the following set:

$$\tilde{A}_{\mathcal{F}} = \left\{ x \in X : |f(x)| \leq \sup_A |f|, \text{ for all } f \in \mathcal{F} \right\}.$$

**Definition(1):**

Let  $E$  be a Banach space and let  $U$  be an open subset of  $E$ . We say that  $U$  is:

- (a)  $\mathcal{P}_{(a+\varepsilon)}(E)$ -convex if  $\tilde{A}_{\mathcal{P}(E)} \cap U$  is  $U$ -bounded, for every  $U$ -bounded subset  $A$ ;
- (b) strongly  $\mathcal{P}_{(a+\varepsilon)}(E)$ -convex if  $\tilde{A}_{\mathcal{P}(E)} \subset U$  and is  $U$ -bounded, for every  $U$ -bounded subset  $A$ ;
- (c)  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex if  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \cap U$  is  $U$ -bounded, for every  $U$ -bounded subset  $A$ ;
- (d) strongly  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex if  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$  and is  $U$ -bounded, for every  $U$ -bounded subset  $A$ ;
- (e)  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex if  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(U)} \cap U$  is  $U$ -bounded, for every  $U$ -bounded subset  $A$ ;

The following lemma shows that the notions of (strongly)  $\mathcal{P}_{(a+\varepsilon)}(E)$ -convex and (strongly)  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex set coincide.

**Lemma (2):**

Let  $A$  be a bounded subset of  $E$ . Then  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} = \tilde{A}_{\mathcal{P}(E)}$ .

**Proof:**

Since  $\mathcal{P}(E) \subset \mathcal{H}_{(a+\varepsilon)}(E)$ , we have that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset \tilde{A}_{\mathcal{P}(E)}$ . Now let us suppose that there exists  $a \in \tilde{A}_{\mathcal{P}(E)}$  such that  $a \notin \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ . Let  $f \in \mathcal{H}_{(a+\varepsilon)}(E)$  be such that  $|f(a)| > \sup_A |f|$ , since

$\tilde{A}_{\mathcal{P}(E)}$  is bounded and  $\mathcal{P}(E)$  is dense in  $\mathcal{H}_{(a+\varepsilon)}(E)$  for the  $\tau_{(a+\varepsilon)}$  topology. Given  $\varepsilon > 0$  there exists  $P \in \mathcal{P}(E)$  such that  $\sup_{\tilde{A}_{\mathcal{P}(E)}} |f - P| < \frac{\varepsilon}{2}$ . In particular we have that

$$\sup_A |p| \leq \sup_A |p - f| + \sup_A |f| < \frac{\varepsilon}{2} + \sup_A |f|.$$

Finally we get that  $|f(a)| \leq |f(a) - p(a)| + |p(a)| < \frac{\varepsilon}{2} + \sup_A |p| < \varepsilon + \sup_A |f|$ , for all  $\varepsilon > 0$ , which is a contradiction.

**Lemma (3):**

If  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ , for every  $U$ -bounded subset  $A$ , then  $U$  is strongly  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex.

**Proof:**

We follow ideas of [9]. Let  $A$  be a  $U$ -bounded subset.

We must show that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$  is  $U$ -bounded. Since it is clear that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$  is bounded, it remains to show that there exists  $\varepsilon > 0$  such that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} + B(0, \varepsilon) \subset U$ . Let  $\varepsilon > 0$  be such that  $A + B(0, \varepsilon)$  is  $U$ -bounded. Then  $(A + B(0, \varepsilon))^{\wedge}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ . Let  $y \in \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ ,  $t \in B(0, \varepsilon)$  and  $0 < \theta < 1$ . Then for each  $f \in \mathcal{H}_{(a+\varepsilon)}(E)$  we have that

$$|f(y + \theta t)| \leq \sum_{m=0}^{\infty} \theta^m |p_t^m(f)(y)| \leq \sum_{m=0}^{\infty} \theta^m \sup_A |p_t^m(f)| \leq (1 - \theta)^{-1} \sup_{A+B(0,\varepsilon)} |(f)|,$$

where the second inequality follows because  $p_t^m(f) \in \mathcal{H}_{(a+\varepsilon)}(E)$  and  $y \in \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ . The third inequality follows by applying [10], with  $t \in B(0, \varepsilon)$  and  $r = 1$ .

Next we apply the above inequality to  $f^n$ , take  $n$ -th roots and let  $n \rightarrow \infty$  to get that  $|f(y + \theta t)| \leq \sup_{A+B(0,\varepsilon)} |f|$ , that is,  $y + \theta t \in (A + B(0, \varepsilon))^{\wedge}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ . By letting  $\theta \rightarrow 1$  we have that  $y + t \in U$ , and the conclusion follows.

**Lemma (4):**

Let  $\mathcal{F} \in \mathcal{H}_{(a+\varepsilon)}(E)$  be a family with the property that the function  $x \mapsto f(\lambda x)$  is an element of  $\mathcal{F}$ , for every  $f \in \mathcal{F}$  and  $|\lambda| \leq 1$ . Let  $A \subseteq E$  be a balanced subset. Then  $\tilde{A}_{\mathcal{F}}$  is balanced.

**Proof:**

Let  $f \in \mathcal{F}$ . For each  $\lambda \in \mathbb{C}$  such that  $|\lambda| \leq 1$ , let  $f_{\lambda} \in \mathcal{F}$  be such that  $f_{\lambda}(x) = f(\lambda x)$ , for all  $x \in E$ . Let  $y \in \tilde{A}_{\mathcal{F}}$ . Then  $|f(\lambda y)| = |f_{\lambda}(y)| \leq \sup_A |f_{\lambda}| \leq \sup_A |f|$ , proving that  $\lambda y \in \tilde{A}_{\mathcal{F}}$ , and hence  $\tilde{A}_{\mathcal{F}}$  is balanced.

It is clear that strongly  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex open subsets are always  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex. Also, it is easy to see that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \cap U$ , and hence we have that  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex open subsets are always  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex. The next Proposition shows that if  $U$  is balanced, then all these notions coincide.

### Propositions (5):

Let  $U \subset E$  be a balanced open subset. Then the following conditions are equivalent.

- $U$  is strongly  $\mathcal{P}_{(a+\varepsilon)}(E)$ -convex.
- $U$  is strongly  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex.
- $U$  is  $\mathcal{P}_{(a+\varepsilon)}(E)$ -convex.
- $U$  is  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex.
- $U$  is  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex.

### Proof:

The implication (a)  $\Leftrightarrow$  (b) and (c)  $\Leftrightarrow$  (d) were proved in Lemma (2). The implication (b)  $\Rightarrow$  (d)  $\Rightarrow$  (e) were commented above.

(e)  $\Rightarrow$  (d) we show that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(U)} = \tilde{A}_{\mathcal{P}(E)} \cap U = \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \cap U$ , for every  $U$ -bounded subset  $A$ , and then conclude that  $U$  is  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex. Let  $y \in \tilde{A}_{\mathcal{P}(E)} \cap U$  and we show that  $y \in \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(U)}$ . Let  $f \in \mathcal{H}_{(a+\varepsilon)}(E)$  fixed. Since  $U$  is  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex, the set  $B = \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(U)} \cup \{y\}$  is  $U$ -bounded, and since  $U$  is balanced, given  $\varepsilon > 0$ , there is  $P \in \mathcal{P}(E)$  such that  $\sup_B |f - P| < \frac{\varepsilon}{2}$ .

Then

$$|f(y)| \leq |f(y) - P(y)| + |P(y)| < \frac{\varepsilon}{2} + \sup_A |P|.$$

On other hand:

$$\sup_A |P| \leq \sup_A |f - P| + \sup_A |f| < \frac{\varepsilon}{2} + \sup_A |f|.$$

And finally we get that  $|f(y)| < \varepsilon + \sup_A |f|$ , for all  $\varepsilon > 0$ , which implies that  $y \in \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(U)}$ .

(d)  $\Rightarrow$  (b) let  $A$  be an  $U$ -bounded subset. By Lemma(3), it suffices to prove that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ . First we assume that  $A$  is balanced. Let  $x \in \tilde{A}_{\mathcal{P}(E)}$ . Define  $D_1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$  and  $D = \{\lambda \in D_1 : \lambda x \in U\}$ . Then  $D$  is a disk centered at the origin because  $U$  is balanced, and  $D$  is an open subset of  $D_1$  because  $U$  is open. Let  $\varepsilon > 0$  be such that  $\tilde{A}_{\mathcal{P}(E)} \cap U + B(0, \varepsilon) \subset U$ . Let  $\lambda \in D_1, \lambda x \in U$ , and let  $\mu \in D_1$  be such that  $|\mu - \lambda| \|x\| < \varepsilon$ . Then  $\lambda x \in \tilde{A}_{\mathcal{P}(E)} \cap U$  because  $x \in \tilde{A}_{\mathcal{P}(E)}$  and  $\tilde{A}_{\mathcal{P}(E)}$  is balanced by Lemma(4). Furthermore  $\|\mu x - \lambda x\| < \varepsilon$ , hence

$\mu x \in \tilde{A}_{\mathcal{P}(E)} \cap U + B(0, \varepsilon)$ , and therefore  $\mu x \in U$ .

This implies that any point on the boundary of  $D$  belongs to  $D$ , and  $D$  is an open and closed subset of  $D_1$ , and therefore  $D = D_1$ . It follows that  $x = 1x \in U$ . Since this holds for any  $x \in \tilde{A}_{\mathcal{P}(E)}$ , we have proved that  $\tilde{A}_{\mathcal{P}(E)} \subset U$ .

If  $A$  is not balanced, we consider  $B = ba(A)$ , the balanced hull of  $A$ .

It follows by [5] that,  $B$  is a balanced  $U$ -bounded subset. Then we apply the arguments above and get that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset \tilde{B}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ .

### Corollary (6):

Let  $A$  be a bounded subset of  $E$ . Then  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} = \tilde{A}_{\mathcal{P}(E)}$ .

### Proof:

Since  $\mathcal{P}(E) \subset \mathcal{H}_{(a+\varepsilon)}(E)$ , we have that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset \tilde{A}_{\mathcal{P}(E)}$ . For  $a \in \tilde{A}_{\mathcal{P}(E)}$  such that  $a \notin \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$ , let  $f_j \in \mathcal{H}_{(a+\varepsilon)}(E)$  be such that  $\sum_{j=1}^n |f_j(a)| > \sup_A \sum_{j=1}^n |f_j|$ . Since  $\tilde{A}_{\mathcal{P}(E)}$  is bounded and  $\mathcal{P}(E)$  is dense in  $\mathcal{H}_{(a+\varepsilon)}(E)$  for the  $\tau_{(a+\varepsilon)}$  topology. Given  $\varepsilon > 0$  there exists  $P^j \in \mathcal{P}(E)$  such that  $\sup_{\tilde{A}_{\mathcal{P}(E)}} |\sum_{j=1}^n f_j - \sum_{j=1}^n P^j| < \frac{\varepsilon}{2}$ .

We get in particular,

$$\begin{aligned} \sup_A \sum_{j=1}^n |p^j| &\leq \sup_A \sum_{j=1}^n |p^j - f_j| \leq \sup_A \sum_{j=1}^n |f_j| \\ &< \frac{\varepsilon}{2} + \sup_A \sum_{j=1}^n |f_j|. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{j=1}^n |f_j(a)| &\leq \sum_{j=1}^n |f_j(a) - p^j(a)| + \sum_{j=1}^n |p^j(a)| \\ &< \frac{\varepsilon}{2} + \sup_A \sum_{j=1}^n |p^j|, \quad \text{for } \varepsilon > 0. \end{aligned}$$

which is a contradiction.

### Corollary (7):

Let  $\mathcal{F} \in \mathcal{H}_{(a+\varepsilon)}(E)$  be a family with the property that the function  $x \mapsto f((\lambda_1 + \lambda_2 + \dots + \lambda_n)x)$  is an element of  $\mathcal{F}$ , for every  $f \in \mathcal{F}$  and  $|\lambda_1 + \lambda_2 + \dots + \lambda_n| \leq 1$ . Let  $A \subseteq E$  be a balanced subset. Then  $\tilde{A}_{\mathcal{F}}$  is balanced.

### Proof:

Let  $f \in \mathcal{F}$ , for each  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$  such that  $|\lambda_1 + \dots + \lambda_n| \leq 1$ , let  $f_{(\lambda_1 + \dots + \lambda_n)} \in \mathcal{F}$  be such that



$$f_{(\lambda_1 + \dots + \lambda_n)}(x) = f((\lambda_1 + \dots + \lambda_n)x), \quad \text{for any } x \in E. \text{ Let } y \in \tilde{A}_{\mathcal{F}}. \text{ Then}$$

$$|f((\lambda_1 + \lambda_2 + \dots + \lambda_n)y)| = |f_{(\lambda_1 + \dots + \lambda_n)}(y)|$$

$$\leq \sup_A |f_{(\lambda_1 + \lambda_2 + \dots + \lambda_n)}| \leq \sup_A |f|,$$

proving that  $(\lambda_1 + \dots + \lambda_n)y \in \tilde{A}_{\mathcal{F}}$ , and hence  $\tilde{A}_{\mathcal{F}}$  is balanced.

Next we give some examples.

#### Example (8):

Let  $P \in \mathcal{P}(^m E; F)$  and let  $U = \{x \in E: \|P(x)\| < 1\}$ . Then  $U$  is a balanced  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex open set.

#### Proof:

Clearly  $U$  is a balanced open set. Let  $A$  be an  $U$ -bounded subset of  $U$ . Let  $\varepsilon > 0$  denote the distance from  $A$  to the boundary of  $U$ , and let  $r = \sup_{x \in A} \|x\|$ . If  $x \in A$  and  $1 \leq \lambda < 1 + \frac{\varepsilon}{r}$ , then  $\|\lambda x - x\| = |\lambda - 1|\|x\| < \varepsilon$ , hence  $\lambda x \in U$ , and therefore

$$\|P(x)\| = \left\| P\left(\frac{1}{\lambda}\lambda x\right) \right\| = \lambda^{-m} \|P(\lambda x)\| < \lambda^{-m}.$$

Taking in the right-hand side the infimum over all  $\lambda$  such that  $1 \leq \lambda < 1 + \frac{\varepsilon}{r}$ , we conclude that

$$\|P(x)\| \leq c := \left(1 + \frac{\varepsilon}{r}\right)^{-m} < 1 \text{ for every } x \in A.$$

Let us show that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ . Let  $y \in \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$  and  $\varphi \in F'$ . Then  $\varphi \circ P \in \mathcal{H}_{(a+\varepsilon)}(E)$  and hence

$$|\varphi \circ P(y)| \leq \sup_A |\varphi \circ P|.$$

Now

$$\|P(y)\| = \sup_{\varphi \in B_{F'}} |\varphi(P(y))| \leq \sup_{\varphi \in B_{F'}} \sup_{x \in A} |\varphi(P(x))| \leq \sup_{x \in A} \|P(x)\| < 1,$$

and hence  $y \in U$ .

This shows that  $\tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)}$  is  $U$ -bounded, because if  $0 < c < 1$ , then every bounded subset of  $\{x \in E: \|P(x)\| \leq c\}$  is  $U$ -bounded. Hence  $U$  is strongly  $\mathcal{H}_{(a+\varepsilon)}(E)$ -convex by Lemma (3). Finally  $U$  is  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex by Proposition (5).

#### Corollary (9):

Let  $P \in \mathcal{P}(^m E)$  and let  $U = \{x \in E: |P(x)| < 1\}$ . Then  $U$  is a balanced  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex open set.

#### Corollary (10):

Let  $A \in \mathcal{L}(E_1, \dots, E_m; F)$  and  $E = E_1 \times \dots \times E_m$ . Then

$$U = \{(x_1, \dots, x_m) \in E: \|A(x_1, \dots, x_m)\| < 1\}$$

is a balanced  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex open set.

#### Corollary (11):

Let  $U = \{(x, \lambda) \in E \times \mathbb{C}: \|\lambda x\| < 1\}$ . Then  $U$  is a balanced  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex open set. In [13], B. Tsirelson constructed a reflexive Banach space  $X$ , with an unconditional Schauder basis, that does not contain any

subspace which is isomorphic to  $c_0$  or to any  $\ell_p$ . R. A lencar, R. Aron and S. Dineen proved in [1] that  $\mathcal{P}_f(^m X)$  is norm-dense in  $\mathcal{P}(^m X)$ , for all  $m \in \mathbb{N}$ . Inspired by this result, we will say that a Banach space  $E$  is a Tsirelson-like space if  $E$  is reflexive and  $\mathcal{P}_f(^m E)$  is norm-dense in  $\mathcal{P}(^m E)$ , for all  $m \in \mathbb{N}$ .

We have The following theorem.

#### Theorem (12):

Let  $E$  be a Tsirelson-like space, and let  $U$  be a balanced  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex open subset of  $E$ . Then the spectrum of  $\mathcal{H}_{(a+\varepsilon)}(U)$  is identified with  $U$ .

#### Proof:

Since  $U$  is balanced and  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex, it follows by Proposition (5) that  $U$  is strongly  $\mathcal{H}_{(a+\varepsilon)}(U)$ -convex. Now we follow the ideas of [11]. Let  $T: \mathcal{H}_{(a+\varepsilon)}(U) \rightarrow \mathbb{C}$  be a continuous homomorphism. Then there exists  $C > 0$  and a  $U$ -bounded subset  $A \subset U$  such that

$$|T(f)| \leq C \sup_A |f|, \quad \text{for all } f \in \mathcal{H}_{(a+\varepsilon)}(U).$$

Since  $T$  is multiplicative, we have that

$$|T(f)|^n = |T(f^n)| \leq C \sup_A |f|^n \text{ for every } n \in \mathbb{N}.$$

Taking  $n$ -th roots and making  $n \rightarrow \infty$  we conclude that actually  $C = 1$ . Let  $r > 0$  such that  $A \subset B(0, r)$ . In particular, we have that

$$|T(f)| \leq \sup_A |f| \leq \sup_{B(0,r)} |f|, \quad \text{for all } f \in E'.$$

Hence we have that  $T|_{E'} \in E'' = E$ , so there exists a unique  $a \in E$  such that  $T(f) = f(a)$ , for all  $f \in E'$ , and hence  $T(P) = P(a)$ , for all  $P \in \mathcal{P}_f(^m E)$ , for all  $m \in \mathbb{N}$ . Since  $\mathcal{P}_f(^m E)$  is norm-dense in  $\mathcal{P}(^m E)$ , for all  $m \in \mathbb{N}$ , it follows that  $T(P) = P(a)$ , for all  $P \in \mathcal{P}(E)$ . Then we have that  $|P(a)| = |P(f)| \leq \sup_A |P|$ , for all  $P \in \mathcal{P}(E)$ , which implies that  $a \in \tilde{A}_{\mathcal{P}(E)} = \tilde{A}_{\mathcal{H}_{(a+\varepsilon)}(E)} \subset U$ . Since  $U$  balanced, we have that  $\mathcal{P}(E)$  is  $\tau_{(a+\varepsilon)}$ -dense in  $\mathcal{H}_{(a+\varepsilon)}(E)$ , and then we conclude that  $T(f) = f(a)$ , for all  $f \in \mathcal{H}_{(a+\varepsilon)}(E)$ , proving the Theorem.

#### Definition (13):

Let  $E$  be a Banach space and let  $U$  be an open subset of  $E$ . We say that  $U$  is a  $\mathcal{H}_{(a+\varepsilon)}(U)$ -domain of holomorphy if there are no open sets  $V$  and  $W$  in  $E$  with the following properties:

- (a)  $V$  is connected and not contained in  $U$ ;
- (b)  $\emptyset \neq W \subset U \cap V$ ;
- (c) for each  $f \in \mathcal{H}_{(a+\varepsilon)}(U)$  there exists  $\tilde{f} \in \mathcal{H}(V)$  such that  $\tilde{f} = f$  on  $W$ .

The following corollary is the announced result for balanced  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ -domains of holomorphy.

**Corollary (14):**

Let  $E$  be a Tsirlson-like space, and let  $U$  be a balanced  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ -domain of holomorphy in  $E$ . Then the spectrum of  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$  is identified with  $U$ .

The following result is a consequence of Corollary (14). It says that under the hypotheses of Corollary(14), every proper finitely generated ideal of  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$  has a common zero.

**Theorem (15):**

Let  $E$  be a Tsirlson-like space. Let  $U \subset E$  be a balanced  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ -domain of holomorphy. Then given  $f_1, \dots, f_n \in \mathcal{H}_{(\alpha+\varepsilon)}(U)$  without common zeros, we can find  $g_1, \dots, g_n \in \mathcal{H}_{(\alpha+\varepsilon)}(U)$  such that  $\sum_{i=1}^n f_i g_i = 1$ .

In [3], S. Banach proved that two compact metric spaces  $X$  and  $Y$  are homomorphic if and only if the Banach algebras  $\mathcal{C}(X)$ ,  $\mathcal{C}(Y)$  are isometrically isomorphic. M.H. Stone, in [12], generalized this result to arbitrary compact Hausdorff topological spaces, the well-known Banach –Stone theorem.

### Algebras of holomorphic germs

In [14], D.M. Vieira presents similar results for algebras of holomorphic functions of bounded type, using results on the spectrum of such algebras. More specifically, let  $E$  and  $F$  be reflexive spaces, one of them a Tsirelson-like space, and let  $U \subset E$  and  $V \subset F$  be absolutely convex open subsets. Then it is shown that the algebras  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$  and  $\mathcal{H}_{(\alpha+\varepsilon)}(V)$  are topologically isomorphic, if and only if there is a special type of holomorphic mappings between  $U$  and  $V$ . To show these results we use the characterization of the spectra of  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$  with  $U$  due to J. Mujica, [11]. Now we are going to generalize this result to balanced  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ -domains of holomorphy, using the characterization of the spectrum of  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$ .

Let  $E$  and  $F$  be Banach spaces, and  $U \subset E$  and  $V \subset F$  be open subsets of  $E$  and  $F$ , respectively. We denote by  $\mathcal{H}_{(\alpha+\varepsilon)}(V, U)$  the set of all holomorphic mappings  $\varphi : V \rightarrow E$ , with  $\varphi(V) \subset U$ , such that  $\varphi$  maps  $V$ -bounded subsets into  $U$ -bounded subsets.

**Theorem (16):**

Let  $E$  and  $F$  be reflexive Banach spaces, one of them a Tsirelson-like space. Let  $U \subset E$  and  $V \subset F$  be balanced  $\mathcal{H}_{(\alpha+\varepsilon)}$ -domains of holomorphy. Then the following conditions are equivalent.

- There exists a bijective mapping  $\varphi : V \rightarrow U$  such that  $\varphi \in \mathcal{H}_{(\alpha+\varepsilon)}(V, U)$  and  $\varphi^{-1} \in \mathcal{H}_{(\alpha+\varepsilon)}(U, V)$ .
- the algebras  $\mathcal{H}_{(\alpha+\varepsilon)}(U)$  and  $\mathcal{H}_{(\alpha+\varepsilon)}(V)$  are topologically isomorphic.

In [14] it is shown that if  $K \subset E$  and  $L \subset F$  are absolutely convex compact subsets of Tsirelson-like spaces, then the algebras  $\mathcal{H}(K)$  and  $\mathcal{H}(L)$  are topologically isomorphic if and only if  $K$  and  $L$  are biholomorphically equivalent.

The key to the proof of such result is a theorem of Banach-Stone type between algebras of holomorphic functions of bounded type [14]. We are going to present a generalization of this result to greater class of compact sets, using Theorem (15).

Let  $E$  be a Banach space, and let  $K \subset E$  be a compact subset. We define  $\mathcal{H}(K)$  to be the algebra of all functions that are holomorphic on some open neighborhood of  $K$ . The elements of  $\mathcal{H}(K)$  are called germs of holomorphic functions. We endow  $\mathcal{H}(K)$  with the locally convex inductive limit of the locally convex algebras  $(\mathcal{H}(U), \tau_\omega)$ , where  $U$  varies among the open subsets of  $E$  such that  $K \subset U$ . If  $U_n = K + B(0, \frac{1}{n})$ , for all  $n \in \mathbb{N}$ , then it is easy to see that:

$$(\mathcal{H}(K), \tau_\omega) = \varinjlim_{n \in \mathbb{N}} \mathcal{H}_{(\alpha+\varepsilon)}(U_n).$$

**Definition (17):**

Let  $E$  be a Banach space, let  $K$  be a compact subset of  $E$  and let  $m \in \mathbb{N}$ . We say that  $K$  is  $\mathcal{P}^{(m)}E$ -convex if  $K = \bar{K}_{\mathcal{P}^{(m)}E}$ .

Before we present examples of balanced  $\mathcal{P}^{(m)}E$ -convex compact sets, we shall need the next lemma. If  $A$  is a subset of a Banach space, we denote by  $\bar{\Gamma}(A)$  the closed, absolutely convex hull of  $A$ .

**Lemma(18):**

Let  $E$  be a Banach space and let  $A$  be a bounded subset of  $E$ . Then

$$\hat{A}_{\mathcal{P}_f^{(m)}E} \subset \bar{\Gamma}(A), \text{ for all } m \in \mathbb{N}.$$

**Proof:**

Let  $y \notin \bar{\Gamma}(A)$ . By the Hahn-Banach Theorem, there exists  $\varphi \in E'$  such that  $|\varphi(y)| > \sup_{x \in \bar{\Gamma}(A)} |\varphi(x)|$ .

Hence  $|\varphi^m(y)| > \sup_{x \in \bar{\Gamma}(A)} |\varphi^m(x)| \geq \sup_A |\varphi^m|$ , which shows that  $y \notin \hat{A}_{\mathcal{P}_f^{(m)}E}$ .

**Example (19):**

Every absolutely convex compact subset of Banach space  $E$  is  $\mathcal{P}^{(m)}E$ -convex, for all  $m \in \mathbb{N}$ .

**Proof:**

Let  $K \subset E$  be an absolutely convex compact set. Since  $\mathcal{P}_f(mE) \subset \mathcal{P}(mE)$ , we have that  $\bar{K}_{\mathcal{P}(mE)} \subset \bar{K}_{\mathcal{P}_f(mE)} \subset \bar{\Gamma}(K) = K$ , where the last inclusion follows by Lemma(18).

**Example (20):**

Let  $E$  be a Banach space, and  $L \subset E$  be a compact, balanced and  $\mathcal{P}(mE)$ -convex set. Let  $P \in \mathcal{P}(mE)$ . Then  $K = \{x \in L: |P(x)| \leq 1\}$  is compact, balanced and  $\mathcal{P}(mE)$ -convex.

**Theorem (21):**

Let  $E$  be a Banach space and let  $K$  be a compact, balanced and  $\mathcal{P}(mE)$ -convex subset of  $E$ , for some  $m \in \mathbb{N}$ . Let  $U$  be an open subset of  $E$  such that  $K \subset U$ . Then there exists an open set  $V \subset E$  which is balanced and  $\mathcal{H}_{(a+\varepsilon)}(V)$ -convex, such that  $K \subset V \subset U$ .

**Proof:**

We begin with a slight modification of [10]. If  $\bar{\Gamma}(K) \subset U$ , then we take  $V = \bar{\Gamma}(K) + B(0, \varepsilon)$ , where  $\varepsilon$  is such that  $\bar{\Gamma}(K) + B(0, \varepsilon) \subset U$ . If  $\bar{\Gamma}(K)$  is not contained in  $U$ , then for each  $a \in \bar{\Gamma}(K) \setminus U$  there is  $P \in \mathcal{P}(mE)$  such that  $\sup_K |P| < 1 < |P(a)|$ . Since  $\bar{\Gamma}(K) \setminus U$  is compact, we can find polynomials  $P_1, \dots, P_k \in \mathcal{P}(mE)$  such that

$$\bar{\Gamma}(K) \setminus U \subset \bigcup_{j=1}^k \{x \in E: |P_j(x)| > 1\}.$$

Now it is easy to see that  $\{x \in \bar{\Gamma}(K): |P_j(x)| \leq 1, \text{ for } j = 1, \dots, k\} \subset U$ .

Next we follow the arguments of [10], finding a positive number  $\delta > 0$  such that:

$$V = (\bar{\Gamma}(K) + B(0, \varepsilon)) \cap \{x \in E: |P_j(x)| < 1, \text{ for } j = 1, \dots, k\}.$$

Now  $V$  is balanced and  $\mathcal{H}_{(a+\varepsilon)}(V)$ -convex, by Corollary (9).

Let  $E$  and  $F$  be Banach spaces, and let  $K \subset E$  and  $L \subset F$  be compact subsets. We say that  $K$  and  $L$  are biholomorphically equivalent if there exist open subsets  $U \subset E$  and  $V \subset F$  with  $K \subset U$  and  $L \subset V$  and a biholomorphic mapping  $\varphi: V \rightarrow U$  such that  $\varphi(L) = K$ . The next theorem is the announced result for algebras of holomorphic germs, and generalizes [14].

**Theorem (22):**

Let  $E$  and  $F$  be Tsirelson-like spaces. Let  $K \subset E$  and  $L \subset F$  be balanced compact subsets, such that  $K$  is  $\mathcal{P}(mE)$ -convex, and  $L$  is  $\mathcal{P}(kF)$ -convex, for some  $m, k \in \mathbb{N}$ . Then the following conditions are equivalent.

(a)  $K$  and  $L$  are biholomorphically equivalent.

(b) The algebras  $\mathcal{H}(K)$  and  $\mathcal{H}(L)$  are topologically isomorphic.

**Proof:**

(a)  $\Rightarrow$  (b) [14] applies.

(b)  $\Rightarrow$  (a) We claim that  $\mathcal{H}(K)$  is the inductive limit of a sequence of Fréchet spaces  $\mathcal{H}_{(a+\varepsilon)}(V_n)$ , where each  $V_n$  is balanced and  $\mathcal{P}_b(E)$ -convex (and the same for  $\mathcal{H}(L)$ ). Indeed, let  $U_n = K + B(0; \frac{1}{n})$ , for every  $n \in \mathbb{N}$ . Applying Theorem (21), for each  $n \in \mathbb{N}$  there exists a balanced  $\mathcal{H}_b$ -convex open subset  $V_n$  such that  $K \subset V_n \subset U_n$ . Since  $\mathcal{H}(K) = \lim_{n \in \mathbb{N}} \mathcal{H}_b(U_n)$ , and the inclusion  $\mathcal{H}_{(a+\varepsilon)}(U_n) \hookrightarrow \mathcal{H}_{(a+\varepsilon)}(V_n)$  is continuous, we have that  $\mathcal{H}(K) = \lim_{n \in \mathbb{N}} \mathcal{H}_b(V_n)$ , and our claim is proved. Next we apply the same arguments of (b)  $\Rightarrow$  (a) of [14].

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