Triangular Units Theorem for Areas of Regular Polygons

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Abstract: ***This paper provides a step-by step derivation of a new formula for finding the area of a regular polygon of any side ninscribed in a circle of radius rin terms of triangular units. It introduces the term triangular units and its corresponding formula. Based on the formula for the area of a regular polygon of side n,*** $A_n = \frac{1}{2}nr^2\left(\sin\frac{360°}{n}\right)u^2$ ***the researchers derived the formula for the area of a regular polygon of side nterms of triangular units.*** $A_n = n\, r^2\, \Delta\, u$, ***where*** $\Delta\, u = ½ \sin 360°/n$. ***This new formula is called Triangular Units Theorem for Areas of Regular Polygon.***

Keywords: polygon, regular polygon, triangular sector of a regular n-gon**,** triangular unit of a regular n-gon**,** area of a regular polygon

1. Introduction

Formulas for the areas of triangles, quadrilaterals, and other polygons have been introduced to us in the classrooms terms of square units. It is very convenient to measure areas in terms of square units. Regular polygons cannot be divided into exact number of squares. However, polygons can be conveniently divided into triangular sectors and triangular units.

This paper presents a derivation of a general formula for finding the area of regular n-gons of a finite number of sides *n*, inscribed in acircle of a given radius *r*. The basic figure for finding the area of a regular polygon is the triangle. Using the formula for finding the area of a triangle, A = ½ bh, and with the application of the trigonometric function of angles, the area of a regular polygon of side n can be derived.

The simplest formula for finding the area of a regular polygon is expressed in terms of the apothem and the perimeter of the polygon. An n-gon can be conveniently divided into n triangular sectors. The apothem *a*or the altitude of one triangular sector can be solved using trigonometric functions. However, a more direct formula for finding the area of a regular n-gon would be a great help to students and professional mathematicians alike.

2. Preliminary Concepts

A regular polygon or is a plane figure all of whose sides and interior angles are congruent. A polygon of *n* sidesis called an *n*- gon.The radius of a regular polygon is the distance from the center toany vertex. It is also the radius of the circle that circumscribes it. As shown in Figure 1 below, a regular *n*-gon can be divided into *n* congruent isosceles triangles. In this study, we will designate them as triangular sectors. In this case the pentagon has five of them. In the figure below, the leg of the isosceles triangle is a radius ***r*** of the polygon. The area of a regular polygon is the number of square units it takes to completely fill it [1].To find the area of a regular polygon, we simply multiply the area of a triangular sector by the number of its sides.

The area of a triangular sector is derived in terms of the central angle and the radius of the polygon. A common formula for the area of a regular n-gon is expressed in terms of the apothem and the side of the n-gon or the perimeter of the n-gon.

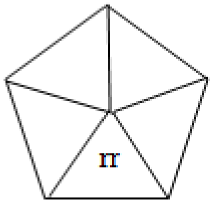

Figure 1: Pentagon with Five Congruent Isosceles Triangles

3. Derivation of Formulas for Areas of Regular Polygons

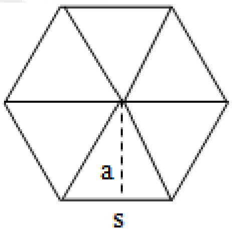

Figure 1: Apothem and Side of a Regular Polygon

Given a regular n-gon with side *s*, apothem *a*, and perimeter$P = n\,s$. Each central angle and a side of the n-gon determines a triangular sector with area $A_\Delta = 1/2\, as$. [1] Since the *n*-gon contains *n* triangular sectors, the area of the n-gon is $A_n = n(A_\Delta) = n\,(1/2\, as)$ or $A_n = 1/2\, a(ns)$. Since P =*n s*, then by substitution, A = 1/2 *a P*. [1,3,5] Using these formulas, the values of *s* and *a*, must either be given or solved for. We can however derive a formula in terms of the radius *r* which will not require solving for *s* or *a*. To show the derivation, consider the figure below.

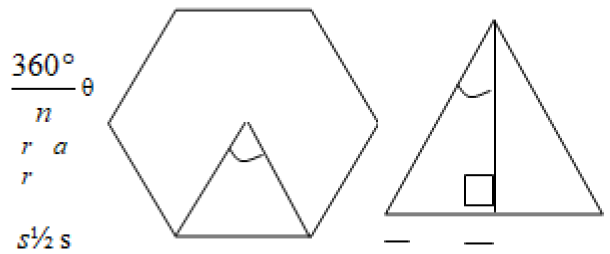

Figure 2: Triangular Sector

Step 1. Finding θ

$$\theta = \frac{1}{2}\left[\frac{360°}{n}\right] = \frac{180°}{n}$$

Step 2. Solving for the apothem in terms of *r* and *n*.

$\cos\theta = \frac{a}{r}$

Substituting the value of θ in step 2 and solving for *a*, we have:

$$a = r\cos\left[\frac{180°}{n}\right]$$

Step 3. Finding s in terms of r and n.

a) $\sin\theta = (1/2\ s)/r$

b) Solving for s, we have $\frac{1}{2}\ s = r\sin\theta$ which results in the following formula:

$$s = 2\,r\sin\left[\frac{180°}{n}\right]$$

Step 4. Finding the area of 1 triangular sector in terms of *r* and *n*. [4]

Using the formula$A_\Delta = \frac{1}{2}as$, substituting the values of *a* and *s*from steps 2 and 3 gives:

$$A_\Delta = 1/2\,[2\,r\sin\frac{180°}{n}]\left[r\cos\frac{180°}{n}\right]$$

The above equation simplifies by substituting the double angle formula for sine.

sin 2θ = 2 sin θ cos θ, with θ = 180°/n

$$\sin 2\frac{180°}{n} = 2\sin\frac{180°}{n}\cos\frac{180°}{n}$$

By substitution, we have:

$A_\Delta = \frac{1}{2}r^2\sin\frac{360°}{n}$ or $A_\Delta = r^2\left[\frac{1}{2}\sin\frac{360°}{n}\right]$

Step 5. Finding the area of a regular n-gon in terms of square units, □ u.

Based on the formula $A_n = n\,(A_\Delta)$, the area of a regular n-gon in terms of square units is given by the formula:

$A_n = n\,r^2\left[\frac{1}{2}\sin\frac{360°}{n}\right]$□ u

Step 6. Finding the area of a triangular unit.

In step 4, we have the formula for the area of a triangular sector,

$A_\Delta = r^2\frac{1}{2}\sin\frac{360°}{n}$.

When r = 1, we have the formula for a triangular unit:

$A_{\Delta u} = \frac{1}{2}\sin\frac{360°}{n}$.

Step 7: Deriving the area of a triangular sector in terms of triangular units, when r >1.

In step 4 we have $A_\Delta = \frac{1}{2}r^2\sin\frac{360°}{n}$or $A_\Delta = r^2\left[\frac{1}{2}\sin\frac{360°}{n}\right]$.

Substituting the value of a

$A_{\Delta u}$ inStep 6 to this formula, we have Theorem 1 below:

Theorem 1: The number of triangular units in a triangular sector of an n-gon is equal to the square of the radius of the n-gon.

$A_\Delta = r^2\Delta$ u

To illustrate this theorem, observe Figure 3 below:

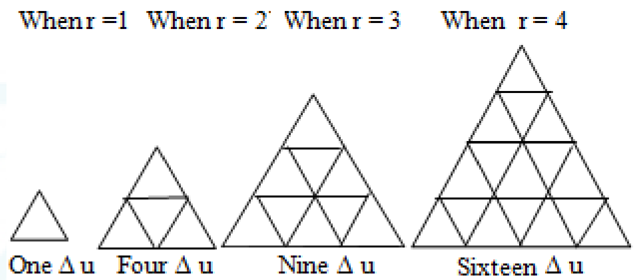

Figure 3: Areas of Triangular Sectors in terms of Triangular Units (Δ u)

Step 8. Finding the area of an n-gon in terms of triangular units.

By multiplying the area of a triangular sector in Lemma 5 by *n*,we come up with the area of a regular n-gon in triangular units. Thus, we have the theorem proven thenext theorem:

Theorem 2: Triangular Units Theorem for Areas of Regular Polygons

The area of a regular n-gon in terms of triangular units (Δ u) is equal to the product of thesquare of the radius and the number of sides *n*.

$$A_n = n\,r^2\Delta \text{ u}$$

To compare the formulas for finding the areas of regular polygons in terms of square units and in terms of triangular units, some examples are given below:

Example 1: Find the area of a regular pentagon, A_5 with r =3 m.

$A_5 = 5\,r^2\frac{1}{2}\left(\sin\frac{360}{5}\right)$ $A_5 = 5\,(3)^2\frac{1}{2}\,(0.951)$ $A_5 = 21.4\ m^2$	$A_5 = 5(3^2)\ \Delta$ m $A_5 = 45\ \Delta$ m

Example 2: Find the area of a regular hexagon ,A_6, with r = 4 cm.

$A_6 = 6\,r^2\frac{1}{2}\left(\sin\frac{360}{6}\right)$ $A_6 = 6\,(4)^2\,\frac{1}{2}(0.866)A_6 = 41.57\ cm^2$	$A_6 = 6(4^2)\ \Delta$ cm $A_6 = 96\ \Delta$ cm

Example 3: Find the area of a regular octagon , $A_{8,}$ with r = 5 m.

$A_8 = 8r^2 \frac{1}{2}\left(\sin\frac{360}{8}\right)$ $A_8 = 8(5)^2\frac{1}{2}(0.707)$ $A_8 = 70.7\text{ m}^2$	$A_8 = 8(5^2)\ \Delta\text{ m}$ $A_8 = 200\ \Delta\text{ m}$

4. Summary

This study introduced a new concept of measuring areas of regular polygons in terms of triangular units instead of the usual square units. The result of the study gives us the following theorems:

Theorem 1: The number of triangular units in a triangular sector of an n-gon is equal to the square of the radius of the n-gon.
$A_\Delta = r^2 \Delta$ u

Theorem 2: Triangular Units Theorem for Areas of Regular Polygons
The area of a regular n-gon in terms of triangular units (Δ u) is equal to the product of the square of the radius and the number of sides *n*.

$$A_n = n\,r^2 \Delta \text{ u}$$

It is observable from the given examples that the area of regular n-gons in terms of triangular units is easier to solve since we get exact answers when r is a whole number. It is also easy to get the equivalent area in terms of square units by substituting the value of a triangular unit in the formula for A_n .

5. Conclusion

There are a number of ways to solve for the areas of regular polygons. The basic approach is to divide the n-gon into n triangular sectors and solve for the area of one triangular sector using the formula for the area of a triangle. Using the same approach, the researcher was able to come up with the idea of measuring the area of regular polygons in terms of triangular units. From the resulting theorem, the researchers make the following conclusions:

1) The formula for the area of a regular n-gon in terms of triangular units gives exact figures when *r* is a whole number.
2) To get the equivalent area of a regular n-gon in terms of square units, we simply substitute the value of a triangular unit in the formula for A_n .
3) The triangular unit for a hexagon is equilateral. All the other triangular units are isosceles triangles.
4) The Triangular Units Theorem for Areas of Regular Polygons has direct applications to engineering, architecture, and the art of tessellation.

6. Acknowledgment

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