On Pricing of an European Call Option in a Financial Market Model with Inertia

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Abstract: ***In aim goal of the present article is to determine the price of call European option in financial market model with inert investment agents, as described in [3], to a mathematical model of a financial market with jumps. This model takes also into account a certain inertial behavior of investors during small time intervals. Next the pricing of an European call option in this type of financial market out for this mathematical model.***

Keywords: Financial market, price model, financial market with jumps and European call option

1. Introduction

In this paper we are interested in a mathematical model which describes a financial market consisting of small investors with different inertial. At a given time some investors are inert and others are active. In their article, Erhan et al. (2003), [3], concentrated themselves on the impact of the inertial behavior of investors to the formation of the price of the underlying financial asset by defining a functional central limit theorem for semi-Markov processes. They presented a simple microstructure model (micro-economy) for the evolution of the price of a financial risky asset where this price is driven by a demand of several small investors with an inertial behavior. The latter mean that these investors are trading in an irregular manner and are inactive most of the time.

Considering the non-Walras approach for the evaluation of the price of an asset and supposing that the price only changes when the market is in disequilibrium, Erhan et al. showed that in case the market consists of a big number of small inactive investors the logarithmic price process of an asset converges to a stochastic process. This process can be written as a stochastic integral with respect to fractional Brownian motion. But this fact poses a problem. Knowing that a stochastic integral which is defined with respect to a fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$ is not a semi martingale, this model may give an opportunity of arbitrage in this market.

Based on the dynamics of the agents behavior, we are going in the first section construct the equation of the price process of the underlying financial asset. The second section will be concentrated on the determination of the equivalent martingale measures before passing on the evaluation of the price of an European call option. In the third section, we will be interested on a comparison of the evolution of the asset price in the context of an investment in the oil sector. The last section, we are going to do some concluding remarks.

2. Model's presentation

We consider a financial market consisting of a certain number of investment agents which trade a single financial asset. The agents are small investors who remain inactive (or inert) in the market during a short time. Let N be the number of agents. Put $A = \{a_1, a_2, \ldots, a_N\}$ where $a_j, 1 \leq j \leq N$, represent the agents in the market. Denote by $S(t)$ the price process of the risky asset, we will first describe in probabilistic terms the dynamics of the individual behavior of the agent.

2.1. Construction of equation of the price process of the underlying financial asset

Let $x = (x_t^a)_{t\geq 0}$ be the mode of trading of agent $a \in A$, who can be active or nonactive in the market during certain intervals. In order to reflect the inertia of agent a, we view the process x as being semi-Markovian on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $E \subseteq \mathbb{R}$. In what follows we write $x = (x_t^a)_{t\geq 0}$. Let us now give an explicit mathematical formula for the process x in terms of a renewal Markov process. Let $\xi_n : \Omega \longrightarrow E \subset \mathbb{R}$ and $T_n : \Omega \longrightarrow \mathbb{R}_+$ be random variables which satisfy $\mathbb{P}$-almost surely:
$0 \leq T_0 \leq T_1 \leq \cdots \leq T_n \leq \cdots$
and
$\mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t \mid \xi_0, \xi_1, \ldots, \xi_n; T_0, T_1, \ldots T_n\} = \mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t \mid \xi_n\},$
for all $n \in \mathbb{N}, j \in E,$ and $t \in \mathbb{R}_+$, where E is the at most countable state of the process x. The process $(\xi, T) = (\xi_n, T_n)$ is called the semi-Markovian renewal process with respect to x (Cocozza-Thivent, 1997). If we consider the process (ξ, T) as being homogeneous in time we get for every $n \geq 0, i, j, i_0, i_1, \ldots, i_{n-1} \in E$ and $t_1, t_2, \ldots, t_n \in \mathbb{R}_+$ the following equality

$$\mathbb{P} = \{\xi_{n+1} = j, T_{n+1} - T_n \leq t \mid \xi_0 = i_0, \xi_1 = i_1, \ldots, \xi_n = i_n; T_1 = t_1, \ldots T_n = t_n\} = \mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t \mid \xi_n = i\} = Q(i, j, t) \quad \mathbf{(1)}$$

The process defined by

$$x_i = \sum_{n\geq 0} \xi_n 1_{[T_n, T_{n+1}[}(t) \quad \mathbf{(2)}$$

is the semi-Markov process which represents the evolution of the trading mode of the agents in the market. This means that the (trading) mode of the agents during the random time interval $[T_n, T_{n+1}]$ is given by the process ξ_n. The distribution of the length $T_{n+1} - T_n$ of the interval $[T_n, T_{n+1}]$ may depend on the sequence $(\xi_n)_{n\in\mathbb{N}}$ between the states ξ_{n+1} and ξ_n. This leads us to the hypothesis that the distribution of the lengths of the intervals of the active and inactive agents is different. In addition to the process which describes the trading mode of the agents we consider a process $\psi = (\psi_t)_{t\geq 0}$ which indicates the level of the volume of the transactions in the market i.e market-wide amplitude process. It is large on time intervals with heavy trading and small on short time intervals of modest trading of the agents. Agent $a \in A$ accumulates assets at the rate $\psi_t x_t^a$. We recall that the random quantity ψ_t indicates the level of the transaction volume at time t.

The portfolio of agent $a \in A$ and the disequilibrium of the market at time $t > 0$ are respectively given by
$\int_0^t \psi_s x_s^a ds$, and $X(t)^N := \sum_{a\in A} \int_0^t \psi_s x_s^a ds$,

Hence, the process $(X(t)^N)_{t\geq 0}$ describes the stochastic evolution of the "disequilibrium of the market". In first times this process will be considered as the only parameter or component which drives the dynamics of the price process of the risky financial asset. In order to describe the distribution of the length of the trading intervals we return to equality (1). The Markov chain $(\xi_n)_{n\in\mathbb{N}}$ on E associated with the renewal process is determined by the transition probabilities matrix $P = (p_{i,j})$ defined by

$$p_{i,j} = \lim_{t\to\infty} Q(i,j,t). \qquad \textbf{(3)}$$

If $p_{i,j} = 0$ for a some pair (i,j), then $Q(i,j,t) = 0$ for all $t \in \mathbb{R}_+$. We can define the quotient $\frac{Q(i,j,t)}{p_{i,j}}$. With this convention, if ξ_{n+1} and ξ_n are given, we can define the function of conditional distribution for the length $T_{n+1} - T_n$ of n^{th} time interval by

$$G(i,j,t) = \frac{Q(i,j,t)}{p_{i,j}} = \mathbb{P}[T_{n+1} - T_n \leq t : \xi_n = i, \xi_{n+1} = j], \textbf{(4)}$$

with $\{Q(i,j,t); i,j,t \in E, t \geq 0\}$ the semi-Markov kernel of x.

Remark 1

1) One way show that, if $Q(i,j,t)$ is of the formation

$$Q(i,j,t) = p_{ij}(1 - e^{-\lambda_i t}), \qquad \textbf{(5)},$$

then x is a homogeneous Markov process.

2) By Proposition (1.9) in Cinlar (1975), it follows that, knowing the Markov chain $(\xi_n)_{n\in\mathbb{N}}$, the sojourn time of a semi-Markovian process are conditionally independent. This means

$$\mathbb{P}\{T_1 - T_0 \leq t_0, \dots, T_{n+1} - T_n \leq t_n : \xi_0, \xi_1, \dots, \xi_{n+1}\} = \prod_{r=0}^n G(\xi_r, \xi_{r+1}, t_r). \qquad \textbf{(6)}$$

From the second point of Remark 1, it follows that the lengths of period of inactivity and activity are independent and identically distributed among each other. As we have said previously, our goal is to analyze the additional effects on the formation of prices of the assets, under the condition that the probability of inactivity during a long time interval is larger that the probability of non-interrupted trading time of an agent. Mathematically speaking such an idea can be expressed by the hypothesis that the distributions of the lengths of the time periods of inactivity of agents have heavy tails, whereas the distributions of the lengths of time intervals of inactivity are small.

In an endeavor to find an approximation result for the dynamics of the trading activity in a financial market containing a certain number of small investors we consider an evolution of the trading process which is faster that the signal of the amplitude of the market. Mathematically we express this by the introduction of a scaling parameter $\epsilon > 0$ and consider the process of the trading mood $x_{t/\epsilon}^a$ (for agent u). For small $\epsilon > 0$, $x_{t/\epsilon}^a$ is called an accelerated semi-Markov process. Observe however, that we do not change the principal qualitative characteristics of the model. The agents remain in a state of inactivity during a longer time than in a state of activity. Mathematically, there is no reason to restrict oneself to processes ψ which are positive. Hence it suffices to suppose that the process ψ is a continuous semi-martingale. Once the process ψ and the processes x^a with $a \in \{a_1, \dots, a_N\}$, we define the process $Y_t^{\epsilon,N}$, the global order rate at time, by

$$Y_t^{\epsilon,N} := \sum_{a\in A} \psi_t x_{t/\epsilon}^a \qquad \textbf{(7)}$$

The work of Erhan et al. (2003), have established the following limit theorem for a financial market with inert investors. This Theorem is announced below.

Theorem 1

Let $\psi = (\psi_t)_{t>0}$ be a semi-martingale on $(\Omega, \mathcal{F}, \mathbb{P}^*)$. Under the hypothesis (2.5) and (2.6) defined in [3] page 6; $\mathbb{E}^*[x_0] = 0$, there exists a positive constant $c > 0$ such that for every $T > 0$, the process of disequilibrium or of deficit of the market $X^{\epsilon,N} = (X(t)^{\epsilon,N})_{t>0}$ defined by

$$X(t)^{\epsilon,N} := \int_0^t Y_t^{\epsilon,N} ds,$$

satisfies

$$\mathcal{L} - \lim_{\epsilon\downarrow 0} \mathcal{L} - \lim_{N\to\infty} \left(\frac{1}{\epsilon^{1-H}\sqrt{NL(\epsilon^{-1})}} X(t)^{\epsilon,N}\right)_{0\leq t\leq T} = \left(c\int_0^t \psi_s dB_s^H\right)_{0\leq t\leq T}, \qquad \textbf{(8)}$$

where the Hurst coefficient of the process of fractional Brownian motion B^H is $H = \frac{3-\alpha}{2} > \frac{1}{2}$.

Remark 2

In economic terms, this theorem says that in a mathematical model of a financial market with several agents of which the periods of inactivity are longer than those of activity and with a certain scale of trading frequency, the logarithmic price process can be approximated in law by a stochastic integral of ψ with respect to fractional Brownian motion with Hurst coefficient $H > \frac{1}{2}$. Such an approximation may lead to an arbitrage opportunity because the process of price is not a semi-martingale. For conditions of existence of arbitrage opportunity the reader is referred to [5] and [7].

If the processes $x^a, a \in A$, are stationary, independent, ergodic and Markovian on E, with average zero, i.e. if the Kernel of the semi-Markovian process is of the form as described in (4), then one can deduce the following result

$$\mathcal{L}\text{-}\lim_{\epsilon\downarrow 0}\mathcal{L}\text{-}\lim_{N\to\infty}\left(\frac{1}{\sqrt{N\epsilon}}X(t)^{\epsilon,N}\right)_{0\le t\le T}=\left(c\int_0^t\psi_s dW_s\right)_{0\le t\le T} \quad \textbf{(9)}$$

where $(W_t)_{t\ge 0}$ is a Brownian motion. This means that if the length of periods of non-activity of an agent is small, there is no arbitrage opportunity because the limit process is a semi-martingale, which in fact is a martingale, provided $\mathbb{E}\left[\int_0^T|\psi_s|^2\right]<\infty$. In our model we are going to consider the fact that the agents remain inactive in small time intervals and that for most of the time they remain active. We also consider the fact that these small moments of inertia of agent may turn certain jumps in the price process of an asset risky. These hypothesis allow us to present our model with inertia.

In the construction of the model we assume that the jumps provoked by the inertia occur according to a law of Poisson. The size of the jumps will be given by real random variables.

Let $N(t)$ be a Poisson process with intensity λ. The variable $N(t)$ indicates the number of times that the jumps occur before or at instant t. Let $Y_1, Y_2, \ldots$ be a sequence of random variables distributed identically with mean $\beta=\mathbb{E}[Y_i]$ which give the sizes of the jumps. We assume the random variables $Y_1, Y_2, \ldots$ are independent of one another and also independent of the Poisson process $N(t)$. One shows that the process defined by

$$Q(t)=\sum_{i=1}^{N(t)}Y_i;\ \forall t\ge 0 \quad \textbf{(10)}$$

is a Poisson process (See [8]); by convention $\sum_{i=1}^{0}Y_i=0$. The jumps of $Q(t)$ are achieved at the same time instants as those of $N(t)$, but the jumps in $N(t)$, are of size 1, while the jumps of $Q(t)$, are of an uncertain size. The first jump has size Y_1, the second jump has size Y_2, …

Denoted by $\varphi^{Y(u)}=\mathbb{E}[e^{uY_i}]$ the characteristic function of the random variable Y_i. Since the random variables $Y_i, i=1,2,\ldots$, are independent and identically distributed the function $\varphi^{Y(u)}$ does not depend in the index i. The moment generating function of the $Q(t)$ is by definition

$$\varphi^{Q(t)^{(u)}}=\mathbb{E}[e^{uQ(t)}]=\mathbb{E}\left[exp\left\{u\sum_{i=1}^{N(t)}Y_i\right\}\right]$$

$$=\exp(-\lambda t)+\sum_{k=1}^{\infty}\mathbb{E}\left[exp\left\{u\sum_{i=1}^{k}Y_i\right\}\right]\frac{(\lambda t)^k}{k!}\exp(-\lambda t)$$

$$=\exp(-\lambda t)\exp(\varphi^{Y(u)}\lambda t)=\exp\left(\lambda t(\varphi^{Y(u)}-1)\right).$$

Because the random variables Y_i tame their values in the finite set $\{y_1,y_2,\ldots,y_M\}, M\in\mathbb{N}$, with $p(y_m)=\mathbb{P}[Y_i=y_m]$ such that $p(y_m)>0$ for all $m=1,\ldots,M$ and $\sum_{m=1}^{M}p(y_m)e^{uy_m}=1$. Then

$$\varphi^{Q(t)^{(u)}}=\prod_{m=1}^{M}exp\{\lambda tp(uy_m)(e^{uy_m}-1)\}.$$

Let $Y_i, i=1,2,\ldots$ be mutually independent and identically distributed random variables. In addition we suppose that they are independent of the process $N(t)$. Under these hypothesis, one can prove that $Q(t)=\sum_{i=1}^{N(t)}Y_i$ is a process with independent increments and mean $\beta\lambda t$.

We notice that in average there are λt jumps in the interval $[0,t]$, the average size of a jump is equal to β, and that the jumps are independent of the size of the jumps. Thus $\mathbb{E}(Q(t))$ is equal to the product $\beta\lambda t$.

2.2. Compensated martingales

Definition 1 Let $(\Omega,\mathcal{N},\mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{N}_t)_{0\le t\le T}$, with $\mathcal{N}_t=\sigma(N(s); 0\le s\le t)$. Let A be a process of finite variation with $A_0=0$ and with total variation which is locally integrable. The unique predictable process $\tilde{A}$, with the property that $A-\tilde{A}$ is a local martingale, is called the compensator of the process A. The local martingale $A-\tilde{A}$ is called the compensated local martingale of the process A.

The existence and the uniqueness of the process $\tilde{A}$ are guaranteed by the theorem of Rao which can be found in ([6]).

From this Rao's theorem, one can prove that the compensated Poisson process $Q(t)-\beta\lambda t$ is a martingale with respect to filtration $(\mathcal{N}_t)_{t\ge 0}$ where $\mathcal{N}_t=\sigma(N(s))_{0\le s\le t}$. Let

$$M(t)=Q(t)-\beta\lambda t, \quad (11)$$

because the time intervals of inertia are shot and the agents remain longer in activity, in our model we incorporate the fact that during the time of activity the evolution of the price process is given by a standard Brownian motion $B(t)_{t\ge 0}$. Taking this hypothesis into account we will introduce in the general equation the following term

$$b(t)dt+\sigma(t)dB(t), \quad (12)$$

Where $b(t)_{t\ge 0}$ and $\sigma(t)_{t\ge 0}$ are processes which are adapted to the filtration generated by $B(t)_{t\ge 0}$. In financial terms $b(t)$ is called the rate of rentability at time instant t, while the coefficient $\sigma(t)$ represents the volatility of the price of the asset during the period of activity. The expression (12) will be considered as the classical term of the general equation of the price process of the risky asset.

The term which describes the price evolution during the small times of inertia is given by

$$\psi(t)dW(t), \quad (13)$$

Where $W(t)$ is a standard Brownian motion as in (9). Since the small inertias induce jumps in the price evolution of the risky asset, we introduce the following term to describe the jumps:

$$\varphi(t)\,dM(t), \quad (14)$$

Where the process $M(t)$ is the compensated martingale defined in (11), while the process $\varphi(t)$ is bounded and predictable with respect to the filtration $\mathcal{N}_t=\sigma(N(s); 0\le s\le t)$. The process $\varphi(t)$ can be interpreted as factor or coefficient of readjustment of the length of the jumps.

2.3. Model of financial market with small inertia and jumps

Consider a financial market constructed on a probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},\mathbb{P})$ jointly with the filtration $(\mathcal{F}_t)_{t\ge 0}$ given by

$\mathcal{F}_t=\sigma(B(s),W(s),N(s))_{0\le s\le t}$.

We assume that there are two assets in the financial market: one of the assets carries the risk and the other one is non-risky. Denote $t \to S^0(t)$ the price evolution of the non-risky asset and $t \to S(t)$ price of risky asset. The process S^0 satisfy stochastic equation

$$\begin{cases} dS^0(t) = S^0(t)r(t)\,dt \\ S^0(0) = 1 \end{cases}, \quad (15)$$

Where $r(t)$ is interest rate on the market. The process S is the solution of the following stochastic differential equation

$$\begin{cases} dS(t) = S(t-)[b(t)dt + \sigma(t)dB(t) + \psi(t)dW(t) + \varphi(t)dM(t)] \\ S(0) = s_0 \end{cases} \quad (16)$$

Where $t \in [0,T]$ and the real number T indicates the finite maturity date. Equation (16) can also be written in the form

$$\begin{cases} dS(t) = S(t-)[(b(t) - \lambda\beta\varphi(t))dt + \sigma(t)dB(t) + \psi(t)dW(t) + \varphi(t)dQ(t)] \\ S(0) = s_0 \end{cases} \quad (17)$$

Since the market contains small time intervals of inertia through the behavior of the agents, the term with $dW(t)$ in equation (16) represents the inertia. The term with $dM(t)$ in the same equation expresses the jumps caused by the inertia of the small agents. The part with pure jumps is given by the term $S(t-)\varphi(t)\,dQ(t)$.

Remark 3
In what follows we suppose that the coefficients σ, ψ, λ, r and φ of the model satisfy the following conditions:

1) The function r is deterministic and bounded and λ is a deterministic constant such that $r(t) \geq 0$ for all $t \in [0,T]$;
2) The process $\sigma(t)$ and $\psi(t)$ are adapted and uniformly bounded in $t \in [0,T]$ and $\omega \in \Omega$;
3) The process $\varphi(t)$ is predictable, continuous and bounded such that
$\varphi(t)\Delta Q(t) > -1$, where $\Delta Q(t) = Q(t) - Q(t-)$.

To show that under the hypothesis of Remark 3, Equation (17) of the model admits a unique solution, let recall the Theorem 2 below. Its proof can be found in [6].

Theorem 2 Let X be a semi-martingale with $X_0 = 0$. Then, there exists a (unique) semi-martingale Z which satisfies $Z(t) = 1 + \int_0^t Z(s-)dX(s)$. The process Z is given by

$$Z(t) = exp\left\{X(t) - \frac{1}{2}[X,X](t)\right\} \times \prod_{0<s\leq t}(1 + \Delta X(s))exp\left\{-\Delta X(s) + \frac{1}{2}(\Delta X(s))^2\right\}. \quad (18)$$

Or, written otherwise,

$$Z(t) = exp\left\{X(t) - \frac{1}{2}[X,X]^c(t)\right\} \times \prod_{0<s\leq t}(1 + \Delta X(s))exp\{-\Delta X(s)\}, \quad (19)$$

Where $[X,X]_t^c$ is the quadratic variation of the continuous part of $X(t)$.

Proposition 1 Under the hypothesis of Remark 3 the equation

$$dS(t) = S_{t^-}[(b(t) - \lambda\beta\varphi(t))dt + \sigma(t)dB(t) + \psi(t)dW(t) + \varphi(t)dQ(t)]$$

Admits a unique solution given by:

$$S(t) = S(0)exp\left[\int_0^t (b(s) - \lambda\beta\varphi(s))ds + \int_0^t \sigma(s)dB(s) + \int_0^t \psi(s)dW(s)\right] \times$$

$$exp\left[-\frac{1}{2}\int_0^t (\sigma^2(s) + \psi^2(s))ds\right]\prod_{0<s\leq t}(1 + \varphi(s)\Delta Q(s)), \quad (20)$$

which can also be written in the form

$$S(t) = S(0)exp\left[\int_0^t (b(s) - \lambda\beta\varphi(s))ds + \int_0^t \sigma(s)dB(s) + \int_0^t \psi(s)dW(s)\right] \times$$

$$exp\left[-\frac{1}{2}\int_0^t (\sigma^2(s) + \psi^2(s))ds + \sum_{0<s\leq t}\ln(1 + \varphi(s)\Delta Q(s))\right] \quad (21)$$

Proof (See [9]).

3. Equivalent martingale measures

In this section we want to present an equivalent measure for the model of the financial market which contains small agents with an inertial behavior. As we remarked earlier the stochastic differential equation describing the price of a risky asset possesses essentially speaking two principal parts: its continuous and its pure jump part. The search of an equivalent martingale measure will consist of determining two processes $Z_1(t)$ for the continuous part and $Z_2(t)$ for the pure jump part in such a way that the process $R(t)S(t)Z(t)$, with $Z(t) = Z_1(t)\ Z_2(t)$ and $R(t) = exp\left[-\int_0^t r(s)ds\right]$, will be a martingale with respect to the original probability measure.

Knowing the two new expressions $N(t) = \sum_{m=1}^M N_m(t)$ and $Q(t) = \sum_{m=1}^M y_m N_m(t)$ respectively for $N(t)$ and $Q(t)$, we will give following a sequence of proposition which will lead to definition of equivalent measure for compound Poisson process.

Proposition 2 Let $(N, \mathbb{P}_{\lambda_1})$ and $(N, \mathbb{P}_{\lambda_2})$ be two Poisson processes on $(\Omega, \mathcal{F}_T)$ with respective intensity λ_1 and λ_2 and jump size respectively a_1 and a_2.

1) If $a_1 = a_2$ then $\mathbb{P}_{\lambda_1}$ is equivalent to $\mathbb{P}_{\lambda_2}$ with Radon-Nikodym density
$$\frac{\mathbb{P}_{\lambda_1}}{\mathbb{P}_{\lambda_2}}\|_{\mathcal{F}_T} = \exp\left[(\lambda_2 - \lambda_1)T - N_T \ln\frac{\lambda_2}{\lambda_1}\right] \quad (22)$$
2) If $a_1 \neq a_2$ then the measures $\mathbb{P}_{\lambda_1}$ and $\mathbb{P}_{\lambda_2}$ are not equivalent.

Proposition 3 Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be compound Poisson processes on $(\Omega, \mathcal{F}_T)$ with Levy measures respectively $\nu^{\mathbb{P}}$ and $\nu^{\mathbb{Q}}$. The measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\nu^{\mathbb{P}}$ and $\nu^{\mathbb{Q}}$ are equivalent. In this case the Radon-Nikodym derivative is given by

$$D_T = \exp[T(\lambda^{\mathbb{P}} - \lambda^{\mathbb{Q}}) + \sum_{s\leq T}\phi(\Delta x_s)], \quad (23)$$

Where $\lambda^{\mathbb{P}} = \nu^{\mathbb{P}}(\mathbb{R})$ and $\lambda^{\mathbb{Q}} = \nu^{\mathbb{Q}}(\mathbb{R})$ are the jump intensities of the two processes and $\phi \equiv \left(\frac{d\nu^{\mathbb{Q}}}{\nu^{\mathbb{P}}}\right)$.
The proof of Propositions 2 and 3 are given in [2].

Proposition 4 Let $(Q, \mathbb{P})$ and $(Q, \tilde{\mathbb{P}})$ be compound Poisson processes on $(\Omega, \mathcal{F}_t)$ with densities λ and $\tilde{\lambda}$. As previously consider the sequence of random variables $Y_1, Y_2, \dots$ which represent the jump sizes of $(Q, \mathbb{P})$ and $(Q, \tilde{\mathbb{P}})$. Let p and $\tilde{p}$ denote the respective distributions of the sizes. For all

$i = 1,2,\ldots$ the variable Y_i takes its values in $\{y_1, y_2, \ldots, y_M\}$.

Then $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent with Radon-Nikodym derivative given by

$$z_2(t) = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \|_{\mathcal{F}_t} = \exp\left[\sum_{m=1}^{M} t(\lambda_m - \tilde{\lambda}_m)\right] \prod_{m=1}^{M} \left(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)}\right)^{N_m(t)}$$

$$= \exp\left(t(\lambda - \tilde{\lambda})\right) \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}, \quad (24)$$

Where $p(y_m) = \mathbb{P}(Y_i = y_m)$ and $\tilde{p}(y_m) = \tilde{\mathbb{P}}(Y_i = y_m)$, $N_m(t)$ the number of jump of size y_m of process $Q(t)$. Moreover, the process $z_2(t)$ in (24) is a $\mathbb{P}$ −martingale with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$.

Proof. Let $y_m \in \{y_1, y_2, \ldots, y_M\}$. We consider $N_m(t)$ as defined above. Then from Proposition 2, we infer that $\mathbb{P}$ and $\tilde{\mathbb{P}}$are equivalent with Radon-Nikodym derivative

$$D_m(t) = \exp\left[(\lambda_m - \tilde{\lambda}_m)t - N_m(t) \ln\frac{\tilde{\lambda}_m}{\lambda_m}\right] =$$

$$\exp\left((\lambda_m - \tilde{\lambda}_m)t\right)\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)} \quad (25)$$

In order to show that $D_m(t)$ is a Radon-Nikodym derivative, it suffices to prove that

1) $dD_m(t) = \frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m} D_m(t-) dM_m(t)$

   où $M_m(t) = N_m(t) - \lambda_m t$.

2) $D_m(t)$ is a martingale with respect to $\mathbb{P}$ and $\mathbb{E}(D_m(t)) = 1$.

Indeed, we define the process X by $X(t) = \frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m} M_m(t)$ which is a martingale with continuous part $X^c(t) = (\tilde{\lambda}_m - \lambda_m)t$ and pure jump part $J(t) = \frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m}$.

Then $[X^c, X^c](t) = 0$, and if there is a jump at time t then

$$\Delta X(t) = \frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m}.$$

Hence

$$1 + \Delta X(t) = \frac{\tilde{\lambda}_m}{\lambda_m}$$

From the definition of $X(t)$ and Equation (25) we remark that the process $D_m(t)$ can be written in the form

$$D_m(t) = \exp\left[X^c(t) - \frac{1}{2}[X^c, X^c](t)\right] \prod_{0<s\le t} (1 + \Delta(s)). \quad (26)$$

According to Theorem 3, we deduce that the process $D_m(t)$ satisfies the equation

$$D_m(t) = 1 + \int_0^t D_m(s-) dX(s).$$

This leads to the equality

$$dD_m(t) = D_m(t-) dX(t) = \frac{\tilde{\lambda}_m - \lambda_m}{\lambda_m} D_m(t-) dM_m(t).$$

Since $X(t)$ is a martingale and $D_m(t-)$ is left continuous then $D_m(t)$ is a martingale. By the fact that $D_m(t)$ is a martingale and $D_m(0) = 1$, we conclude that $\mathbb{E}[D_m(t)] = 1$ for all $t \ge 0$. Hence the process $D_m(t)$ is Radon-Nikodym density on $\mathcal{F}_t$.

For $m \ne n$ the Poisson processes $N_m(t)$ and $N_n(t)$ are independent and in addition have not simultaneous. We have $[D_m(t), D_n(t)] = 0$.

By application of Itô product rule we obtain

$$d(D_1(t)D_2(t)) = D_2(t-)dD_1(t) + D_1(t-)dD_2(t). \quad (27)$$

Because D_1 and D_2 are martingale and the integrals in (27) are left continuous the process D_1D_2 is a martingale.

Since D_1D_2 and D_3 are $\mathbb{P}$ −independent, they do not have common jumps, and therefore it follows that

$$d\left((D_1(t)D_2(t))D_3(t)\right) = D_3(t-)d(D_1(t)D_2(t)) + D_1(t-)D_2(t-)dD_3. \quad (28)$$

The previous arguments also show that the product $D_1D_2D_3$ is a martingale as well. Repeating this procedure M times, we finally see that the product

$$D(t) = D_1(t)D_2(t)D_3(t)\ldots D_M(t)$$

is also a martingale. It follows that

$$D(t) = \prod_{m=1}^{M} D_m(t) = \prod_{m=1}^{M}\left[\exp\left((\lambda_m - \tilde{\lambda}_m)t\right)\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}\right]$$

$$= \exp\left(\sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)t\right)\prod_{m=1}^{M}\left(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)}\right)^{N_m(t)} \quad (29)$$

is a martingale with $D(0) = 1$. Therefore $\mathbb{E}[D(t)] = 1$ for all $t \ge 0$. From equality (29) we conclude that $Z_2(t) = D(t)$ is the Radon-Nikodym derivative relative to the change of measure for the compound Poisson process $Q(t)$. It also shows that the process $Z_2(t)$ is a $\mathbb{P}$ −martingale with respect to the filtration $(\mathcal{F}_t)_{0\le t\le T}$. According to the fact that $Q(t) = \sum_{i=1}^{N(t)} Y_i = \sum_{m=1}^{M} y_m N_m(t)$, we deduce the following equalities:

$$Z_2(t) = \exp\left(\sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)t\right)\prod_{m=1}^{M}\left(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)}\right)^{N_m(t)}$$

$$= \exp\left((\lambda - \tilde{\lambda})t\right)\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}. \quad (30)$$

It is sufficient to take

$$\tilde{\mathbb{P}} = \int_A Z_2(t)\, d\mathbb{P}, \text{ for all } A \in \mathcal{F}_t, \quad t \in [0, T]. \quad (31)$$

We see that the measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent with Radon-Nikodym derivative the process $Z_2(T)$: see the observation in Remark 4 below.

This completes the proof.

Remark 4

Notice that the probability of the event A, i.e. $\tilde{\mathbb{P}}(A)$ in equality (31), does not depend on t as long as $A \in \mathcal{F}_t$. Hence, we may write $\tilde{\mathbb{P}}(A) = \int_A Z_2(T)\, d\mathbb{P}$, $A \in \mathcal{F}_t$. Here we use the fact that the process $Z_2(t)$ is a $\mathbb{P}$ −martingale.

Remark 5

We can easily show that with respect to the measure $\tilde{\mathbb{P}}$, the process $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$. So that the distribution of the jump sizes of $Q(t)$ with respect to the measure $\tilde{\mathbb{P}}$ is equal to $\tilde{p}(Y)$. In the next proposition we assume that the variables $Y_1, Y_2, \ldots$ are still iid, but not discrete anymore.

Proposition 5 Let the hypothesis of Proposition 4 be satisfied, and suppose that the iid. variables $Y_1, Y_2, \ldots$ have densities $f(y)$ and $\tilde{f}(y)$ with respect to $\mathbb{P}$ and $\tilde{\mathbb{P}}$ respectively. Then the measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent with Radon-Nikodym derivative given by the martingale

$$Z_2(t) = \exp\left(\left(\lambda - \tilde{\lambda}\right)t\right)\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}. \quad (32)$$

Proof. The proof of this proposition is similar to the one of Proposition 4.

Knowing the new equivalent measure of the compound Poisson process, will enable us to define the equivalent probability measure for market model with inertia. This means an equivalent measure for the Brownian motions $B(t)$ and $W(t)$, and the compound Poisson process $Q(t)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which the Brownian motions $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ as well as the compound Poisson process $Q(t)$ are defined. Here $Q(t) = \sum_{i=1}^{N(t)} Y_i$, $N(t)$ is a Poisson process with intensity λ and the i.i.d. variables $Y_i, i \in \mathbb{N}$, have $f(y)$ as jump density function. On the space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider the filtration $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \sigma(B(s), W(s), N(s); 0 \leq s \leq t)$.
We know by hypothesis that the Brownian motions $B(t), W(t)$ and the compound Poisson process $Q(t)$ are independent. This fact will be used in Lemma 1 below. Let $\tilde{\lambda}$ be positive real number, and let $\tilde{f}(y)$ be another jump density function with the property that $\tilde{f}(y) = 0 \Leftrightarrow f(y)$ =0. Let $\theta_1(t)$ and $\theta_2(t)$ be two processes which are adapted to $\mathcal{F}_t$ and satisfy the following conditions:

$$\int_0^T \left(\theta_1^2(t) + \theta_2^2(t)\right) dt < \infty$$

and process $Z_1(t)$ is martingale. With this notation we define

$$Z_1(t) = \exp\left[-\int_0^t \theta_1(s)\,dB(s) - \int_0^t \theta_2(s)\,dW(s) - \frac{1}{2}\int_0^t \theta_1^2(s)\,ds - \frac{1}{2}\int_0^t \theta_2^2(s)\,ds\right], \quad (33)$$

$$Z_2(t) = \exp t\left(\lambda - \tilde{\lambda}\right)\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}, \quad (34)$$

and

$$Z(t) = Z_1(t)Z_2(t). \quad (35)$$

Lemma 1 The process $Z(t)$ defined in (35) is a martingale.

In particular $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$.

Proof. It is know that the processes $Z_1(t)$ and $Z_2(t)$ are martingales: see Proposition 4. Since $Z_1(t)$ is continuous and $Z_2(t)$ does not contain a Brownian motion part, we have $[Z_1, Z_2] = 0$. The product rule for Itô calculus shows

$$d\left(Z_1(t)Z_2(t)\right) = Z_2(t-)dZ_1(t) + Z_1(t-)dZ_2(t). \quad (36)$$

Consequently

$$Z_1(t)Z_2(t) = Z_1(0)Z_2(0) + \int_0^t Z_2(s-)\,dZ_1(s) + \int_0^t Z_1(s-)\,dZ_2(s). \quad (37)$$

Since the processes $Z_1(t)$ and $Z_2(t)$ are continuous from the right and have limits from the left, the two integrals in (37) are martingales. Hence the product $Z(t) = Z_1(t)Z_2(t)$ is a martingale. From equality (35) it is clear that $Z(0) = 1$. So we see $\mathbb{E}(Z(t)) = 1$ for every $t \geq 0$, because $Z(t)$ is a martingale. This concludes the proof.

Fix a real number T and define the probability measure $\tilde{\mathbb{P}}$ by:

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\,d\mathbb{P}, \quad \forall A \in \mathcal{F}_T. \quad (38)$$

Then the new measure $\tilde{\mathbb{P}}$ defined in (38) is a probability measure which is equivalent to $\mathbb{P}$ with the process $Z(t)$ as Radon-Nikodym density. The probability measure $\tilde{\mathbb{P}}$ is called the neutral risk measure or adjustment measure.

Theorem 3 With respect to the probability measure $\tilde{\mathbb{P}}$ the processes

$$\tilde{B}(t) = B(t) + \int_0^t \theta_1(s)\,ds$$

and

$$\tilde{W}(t) = W(t) + \int_0^t \theta_2(s)\,ds$$

are Brownian motions, and $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$. The variables $Y_1, Y_2, \ldots$ are independent and identically distributed, with $\tilde{f}(y)$ as density of the distribution of the jump size. In addition, the processes $\tilde{B}(t)$, $\tilde{W}(t)$ and $Q(t)$ are independent.

Proof. The proof of this theorem follows from combination of the Theorem (11.6.7), Theorem (11.6.9) and Theorem (11.6.10) in [8].

Remark 6 Suppose that the compound Poisson process $Q(t)$ has jumps $Y_1, Y_2, \ldots$ which attain their non-zero values in the finite set $\{y_1, y_2, \ldots, y_M\}$ with $p(y_m) = \mathbb{P}[Y_i = y_m]$ such that $p(y_m) > 0$ and $\sum_{m=1}^{M} p(y_m) = 1$.

Let $p(y_1), p(y_2), \ldots, p(y_M)$ be strictly positive quantities with sum is equal to 1. We redefine the new probability measure $\tilde{\mathbb{P}}$ by taking

$$Z_2(t) = \exp\left(t\left(\lambda - \tilde{\lambda}\right)\right)\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)},$$

$$= \exp\left(t\left(\lambda - \tilde{\lambda}\right)\right)\prod_{m=1}^{M}\left(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)}\right)^{N_m(t)} \quad (39)$$

where $N_m(t), 1 \leq m \leq M$, are independent Poisson processes with intensity λ_m.

Corollary 1 With respect to the measure $\tilde{\mathbb{P}}$ as defined in Remark 6, the processes
$\tilde{B}(t) = B(t) + \int_0^t \theta_1(s)ds$ and $\tilde{W}(t) = W(t) + \int_0^t \theta_2(s)ds$ (40)

Are Brownian motions, $Q(t)$ is a Poisson process with intensity $\tilde{\lambda}$ and jump variables $Y_i, i \in \mathbb{N}$, which are i.i.d. and satisfy $\tilde{\mathbb{P}}[Y_i = y_m] = \tilde{p}(y_m)$ for all $i = 1, 2, \ldots, m = 1, 2, \ldots, M$. Moreover, the processes $\tilde{W}(t), \tilde{B}(t)$ and $Q(t)$ are mutually independent. As a consequence for every $m = 1, 2, \ldots, M$ the processes $\tilde{W}(t)$ and $N_m(t)$ are independent as well. Every $N_m(t)$ is a Poisson process with intensity $\tilde{\lambda}_m$. The processes $N_m(t), m \in \mathbb{N}$, are also mutually independent.

Proof. The proof of this Corollary follows from application of the Theorem 3 and Remark 6.

Remark 7 Under the new measure $\tilde{\mathbb{P}}$, we have

1) $N(t) = \sum_{m=1}^{M} N_m(t)$, $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$, with $\tilde{\lambda}$ the intensity of $N(t)$.
2) $\tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$.
3) $\tilde{\beta} = \mathbb{E}[Y_i] = \sum_{m=1}^{M} y_m \tilde{p}(y_m) = \frac{1}{\tilde{\lambda}} \sum_{m=1}^{M} \tilde{\lambda}_m y_m$.
4) The process $Q(t) - \tilde{\beta}\tilde{\lambda}t$ is a martingale with respect to the measure $\tilde{\mathbb{P}}$.

Next we will construct a neutral risk measure. By employing the notation as indicated in the preceding remarks 6 and 7 and also Corollary 1 we know that the price of the risky asset is given by

$$dS(t) = S(t)\left[\left(b(t) - \lambda\beta\varphi(t)\right)dt\right] + \sigma(t)dB(t) + \psi(t)dW(t) + \varphi(t)dQ(t). \quad (41)$$

The discount price process can be written as

$$\tilde{S}(t) = R(t)S(t) \quad (42)$$

with $R(t) = \exp\left(-\int_0^t r(u)du\right)$. By applying Itô's formula to equation (42) we get:

$$d\tilde{S}(t) = S(t)dR(t) + R(t)dS(t)$$
$$= \tilde{S}(t)\left(b(t) - r(t) - \sigma(t)\theta_1(t) - \psi(t)\theta_2(t) - \lambda\beta\varphi(t) + \tilde{\lambda}\tilde{\beta}\varphi(t)\right)dt$$
$$+\tilde{S}(t)\sigma(t)\,d\tilde{B}(t) + \tilde{S}(t)\psi(t)\,d\tilde{W}(t) + \tilde{S}(t-)\varphi(t)\,d\tilde{M}(t) \quad (43)$$

From equality (43) it is clear that the process $\tilde{S}(t)$, representing the discount price process is a martingale if and only if

$$b(t) - r(t) - \sigma(t)\theta_1(t) - \psi(t)\theta_2(t) + \left(\tilde{\lambda}\tilde{\beta} - \lambda\beta\right)\varphi(t) = 0. \quad (44)$$

Equation (44) can be written in the form:

$$r(t) = b(t) - \sigma(t)\theta_1(t) - \psi(t)\theta_2(t) + \left(\tilde{\lambda}\tilde{\beta} - \lambda\beta\right)\varphi(t). \quad (45)$$

It is well-known that

$$\beta = \frac{1}{\lambda}\sum_{m=1}^{M} \lambda_m y_m \text{ and } \tilde{\beta} = \frac{1}{\tilde{\lambda}}\sum_{m=1}^{M} \tilde{\lambda}_m y_m \quad (46)$$

Using (46) the equation in (44) becomes

$$b(t) - r(t) - \sigma(t)\theta_1(t) - \psi(t)\theta_2(t) + \left(\sum_{m=1}^{M} \left(\tilde{\lambda}_m - \lambda_m\right)y_m\right)\varphi(t) = 0. \quad (47)$$

In this way we get an equation with $M+2$ unknowns: $\tilde{\lambda}_1, \dots, \tilde{\lambda}_M$ and θ_1, θ_2. Equation (47) is called the equation of the price of the market risk. It is clear that the equation admits an infinity of solution. This observation implies the existence of several equivalent measures for which the discount price process is a martingale. It follows that the financial market is incomplete.

Remark 8

1) If $\psi(t) = 0$ and $\varphi(t) = 0$ the equation (47) becomes the equation of the market risk for the classic model given by:

$$b(t) - r(t) - \sigma(t)\theta_1(t) = 0 \quad (48)$$

where

$$\theta_1(t) = \frac{b(t) - r(t)}{\sigma(t)},$$

with $\theta_1(t)$ the prime of the market risk.

2) The financial market becomes complete under certain conditions: for example when it is constituted of 4 assets and 2 values of jumps sizes.
3) If the random variables of jumps sizes are not discrete, but have densities $f(y)$ and $\tilde{f}(y)$ with respect to measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ respectively, then it follows that: $\beta = \int_{-1}^{\infty} yf(y)dy$ and $\tilde{\beta} = \int_{-1}^{\infty} y\tilde{f}(y)\,dy$.
The equation (44) becomes

$$b(t) - r(t) - \sigma(t)\theta_1(t) - \psi(t)\theta_2(t) + \left(\int_{-1}^{\infty}\left(\tilde{\lambda}\tilde{f}(y) - \lambda f(y)\right)ydy\right)\varphi(t) = 0. \quad (49)$$

Next we return to the general case and suppose that the unknowns $\tilde{\lambda}_1, \dots, \tilde{\lambda}_M$ and θ_1, θ_2 are chosen in such a way that the equation of the price of market risk (47) is satisfied. Then

$$dS(t) = S(t)[(b(t) - \lambda\beta\varphi(t))dt + \sigma(t)dB(t) + \psi(t)dW(t) + \varphi(t)dQ(t)]$$
$$= S(t)\left[\left(r(t) - \tilde{\lambda}\tilde{\beta}\varphi(t)\right)dt + \sigma(t)\,d\tilde{B}(t) + \psi(t)\,d\tilde{W}(t) + \varphi(t)\,dQ(t)\right], \quad (50)$$

with $\tilde{M}(t) = Q(t) - \tilde{\lambda}\tilde{\beta}t$. The solution of equation (50) is given by

$$S(t) = S(0)\exp\left[\int_0^t \sigma(s)\,d\tilde{B}(s) + \int_0^t \psi(s)\,d\tilde{W}(s) + \int_0^t\left(r(t) - \tilde{\lambda}\tilde{\beta}\varphi(s)\right)ds\right] \times$$
$$\exp\left[-\frac{1}{2}\int_0^t \sigma^2(s)ds - \frac{1}{2}\int_0^t \psi^2(s)ds\right] \times \prod_{0<s\leq t}(1 + \varphi(s)\Delta Q(s)). \quad (51)$$

In the following section, Equation (51) which describes the price process of the risky asset, is going to be very useful for the pricing of the European call option.

4. Pricing of an European call option

In this section we try to evaluate the price of an European call option in the setting of market model with inertia as defined above. Next we determine the bracket of the viable price of such an option.

4.1 European call option

Let a financial market be defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ where

$$\mathcal{F}_t = \sigma(B(s), W(s), N_1(s), \dots, N_M(s), Y_N(s); 0 \leq s \leq t).$$

Consider an European call option defined by an $\mathcal{F}_T$ −measurable random variable h with respect to an underlying risky asset, the price of which is given by equation (51). Let T be the time of maturity of the option and K the price at time of maturity. The random variable h is defined by $h(x) = (x - K)^+$. Te value of the option at time t, with $t \in [0, T]$ is given by

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r(u)du}(S(T) - K)^+ \,\vdots\, \mathcal{F}(t)\right], \quad (52)$$

with

$$S(T) = S(t)\exp\left[\int_t^T \sigma(s)\,d\tilde{B}(s) + \int_t^T \psi(s)\,d\tilde{W}(s) + \int_t^T\left(r(t) - \tilde{\lambda}\tilde{\beta}\varphi(s)\right)ds\right] \times$$
$$\exp\left[-\frac{1}{2}\int_t^T \sigma^2(s)ds - \frac{1}{2}\int_t^T \psi^2(s)ds\right] \times \prod_{t<s\leq T}(1 + \varphi(s)\Delta Q(s)). \quad (53)$$

In what follows later we will calculate the explicit value of an European call option in case that the coefficients r, σ, ψ and φ in equation (51) for $S(t)$ are constant. Before passing to the calculation of the value of the option we announce the following (independence) lemma which can be found in [8].

Lemma 2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ a sub-σ-field of $\mathcal{F}$. Let the random variables $X_1, \dots, X_k$ be $\mathcal{G}$-measurable. Suppose that the random variables $Y_1, \dots, Y_L$ are independent of $\mathcal{G}$. Let $f(x_1, \dots, x_K, y_1, \dots, y_L)$ be a function of the standard variables $x_1, \dots, x_K$ and $y_1, \dots, y_L$ defined in such a way that
$g(x_1, \dots, x_K) = \mathbb{E}[f(x_1, \dots, x_K, y_1, \dots, y_L)]$.
Then
$\mathbb{E}[f(X_1, \dots, X_K, Y_1, \dots, Y_L) \,\vdots\, \mathcal{G}] = g(X_1, \dots, X_K)$.
We introduce the following notation which will be useful in the development of the calculation of the value of an European call option:
$\kappa(\tau, x) = xN(d_+, (\tau, x)) - Ke^{\tau\tau}N(d_-(\tau, x))$,
where

$$d_{\pm} = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]$$

and

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{y} e^{-\frac{1}{2}z^2}\,dz.$$

Remark 9 The function $N(y)$ in the above notation is the cumulative distribution function for the standard normal law. The function $\kappa(\tau, x)$ corresponds to the value of an European call option in a Black-Sholes model based on a geometric Brownian motion with constant volatility σ, with an expiration time which lies τ time units in the future, constant interest rate r, and price at maturity equal to K. In terms of the expectation with respect to $\tilde{\mathbb{P}}$, the function $\kappa(\tau, x)$ can also be written in the form

$$\kappa(\tau, x) = \tilde{\mathbb{E}}\left[e^{\tau r}\left(x\exp\left[-\sigma\sqrt{\tau}Y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right] - K\right)^+\right], \quad (54)$$

where Y is a standard normal random variable with respect to the probability $\tilde{\mathbb{P}}$.

Theorem 4 Suppose that the coefficient σ, ψ and φ are constants in the equation (53) of the price process of the asset $S(t)$. Denote by $t \mapsto S^c(t)$ the continuous part of the price process $t \mapsto S(t), 0 \le t \le T$. Then the value of the European call option on the asset with price process $S^c(t)$ is given by

$$V(t) = \tilde{\mathbb{E}}[e^{-r\tau}(S^c(T) - K)^+ \,\vdots\, \mathcal{F}^{B,W}(t)] = \kappa(\tau, S^c(T)e^{-\tilde{\lambda}\beta\varphi\tau}), \quad (55)$$

where

$$\kappa = x'N(d_2(\tau, x')) - e^{-r\tau}KN(d_1(\tau, x')), \quad (56)$$

$\tau = T - t$,

$$d_1(\tau, x') = \frac{1}{\sqrt{\tau(\sigma^2 + \psi^2)}}\left[\ln\left(\frac{x'}{K}\right) + r\tau - \frac{1}{2}(\sigma^2 + \psi^2)\tau\right]$$
$$= \frac{1}{\sqrt{\tau(\sigma^2 + \psi^2)}}\left[\ln\left(\frac{x}{K}\right) + (-\tilde{\lambda}\beta\varphi + r)\tau - \frac{1}{2}(\sigma^2 + \psi^2)\tau\right], \quad (57)$$

and

$$d_2(\tau, x') = d_1(\tau, x') + \sqrt{\tau(\sigma^2 + \psi^2)}, \quad (58)$$

with

$$x' = xe^{-\tilde{\lambda}\beta\varphi\tau}. \quad (59)$$

In the definition of κ, the notation

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz \quad (60)$$

is used and in (55) the variable x is replaced by $S^c(t)e^{-\tilde{\lambda}\beta\tau}$.

Proof. Since the coefficients σ, ψ, φ and the interest rate r are constants. We have

$$S^c(T) = S^c(t)\exp\left[\sigma\left(\tilde{B}(T) - \tilde{B}(t)\right) + \psi\left(\tilde{W}(T) - \tilde{W}(t)\right) + \left(r - \tilde{\lambda}\beta\varphi\right)(T - t)\right] \times$$
$$\times \exp\left[\left(-\frac{1}{2}\sigma^2 - \frac{1}{2}\psi^2\right)(T - t)\right]. \quad (61)$$

The value of the European call option $V(t)$ at the moment $t \in [0, T]$ on the underlying asset, the price of which is driven by the process $S^c(t)$, the maturity being fixed at time T and with price K at time T reads as

$$V(t) = \tilde{\mathbb{E}}\left[e^{\int_t^T r(u)du}(S^c(T) - K)^+ \,\vdots\, \mathcal{F}^{\tilde{B},\tilde{W}}(t)\right], \quad (62)$$

where $\mathcal{F}^{\tilde{B},\tilde{W}}(t)$ is the σ-field generated by the Brownian motions $\tilde{B}$ and $\tilde{W}$. By carrying the value of $S^c(T)$ from (61) to (62) we infer

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S^c(t)\exp\left(\sigma\left(\tilde{B}(T) - \tilde{B}(t)\right) + \psi\left(\tilde{W}(T) - \tilde{W}(t)\right) + \left(r - \tilde{\lambda}\beta\varphi - \frac{\sigma^2}{2} - \frac{\psi^2}{2}\right)(T - t)\right) - K\right)\right]. \quad (63)$$

The equality in (63) is obtained by the $\mathcal{F}(t)$-measurability of $S^c(t)$ together with fact that the two variables $\tilde{B}(T) - \tilde{B}(t)$ and $\tilde{W}(T) - \tilde{W}(t)$ are independent of $\mathcal{F}^{\tilde{B},\tilde{W}}(t)$. By writing

$$\tau = T - t, \quad Y_B = -\frac{\tilde{B}(T) - \tilde{B}(t)}{\sqrt{\tau}}, \quad Y_W = -\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{\tau}}$$

and choosing $\rho \ge 0$ such that
$\rho^2 = \sigma^2 + \psi^2$
we get

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r\tau}\left(e^{(r-\tilde{\lambda}\beta\varphi)\tau}S(t)\exp\left(-\sigma\sqrt{\tau}Y_B - \psi\sqrt{\tau}Y_W - \frac{\rho^2}{2}\tau\right) - K\right)^+\right].$$

By introducing the variable

$$Y_{BW} = \frac{\sigma Y_B + \psi Y_W}{\sqrt{\sigma^2 + \psi^2}},$$

the value of the option becomes

$$V(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\left(e^{-\tilde{\lambda}\beta\varphi\tau}S(t)\exp\left(-\sqrt{\tau(\sigma^2 + \psi^2)}y - \left(\frac{\sigma^2}{2} + \frac{\psi^2}{2}\right)\tau\right) - Ke^{-r\tau}\right)^+\right]e^{-\frac{1}{2}y^2dy}. \quad (64)$$

To obtain the equality in (64) above we used the fact that Y_{BW} is a standard normal random variable. The integral in (64) is positive because

$$e^{-\tilde{\lambda}\beta\varphi\tau}S(t)\exp\left(-\sqrt{\tau(\sigma^2 + \psi^2)}y - \left(\frac{\sigma^2}{2} + \frac{\psi^2}{2}\right)\tau\right) > Ke^{-r\tau} \quad (65)$$

The inequality in (65) is equivalent to

$$e^{-\tilde{\lambda}\beta\varphi\tau}S(t)\exp\left(-\sqrt{\tau(\sigma^2 + \psi^2)}y - \left(\frac{\sigma^2}{2} + \frac{\psi^2}{2}\right)\tau\right) > Ke^{-r\tau} \quad (65)$$

By taking

$$d_1 = d_1(\tau, s^c(t)e^{-\tilde{\lambda}\beta\varphi\tau}) = \frac{1}{\sqrt{\tau(\sigma^2+\psi^2)}}\left[\ln\left(\frac{S^c(t)}{K}\right) + (r - \tilde{\lambda}\beta\varphi)\tau - \frac{1}{2}(\sigma^2 + \psi^2)\tau\right]$$

and recall that $\rho = (\sigma^2 + \psi^2)^{1/2}$ the integral in (64) can be written as

$$V(t) = \frac{e^{(-\tilde{\lambda}\beta\varphi)\tau} S^c(t)}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp\left(-\frac{1}{2}\left(y + \sqrt{\tau}\rho\right)^2\right) dy - \frac{1}{\sqrt{2\pi}} K e^{-r\tau} \int_{-\infty}^{d_1} e^{-\frac{1}{2}y^2} dy. \quad (67)$$

For brevity in (67) we wrote

$$d_1 = d_1\left(T - t, S^c(t)e^{-\tilde{\lambda}\beta\varphi\tau}\right) = d_1\left(\tau, S^c(t)e^{-\tilde{\lambda}\beta\varphi\tau}\right).$$

A substitution shows the equalities

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \exp\left(\frac{1}{2}\left(y + \sqrt{\tau}\rho\right)^2\right) dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1+\sqrt{\tau}\rho} e^{-\frac{1}{2}z^2} dz = N(d_2) \quad (68)$$

where

$$d_2 = d_1 + \sqrt{\tau}\rho = d_1 + \sqrt{\tau(\sigma^2 + \psi^2)}.$$

We also have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}y^2} dy = N(d_1). \quad (69)$$

Finally using (68) and (69) the equality in (67) can be rewritten as

$$V(t) = S^c(t)e^{-\tilde{\lambda}\beta\varphi\tau} N\left(d_2\left(\tau, S^c(t)e^{-\tilde{\lambda}\beta\varphi\tau}\right)\right) - e^{-r\tau} K N\left(d_1\left(\tau, S^c(t)e^{-\tilde{\lambda}\beta\varphi\tau}\right)\right) = \kappa\left(t, S^c(t)e^{-\tilde{\lambda}\beta\varphi\tau}\right), \quad (70)$$

where, as above $\tau = T - t$, and

$$\kappa(\tau, x') = x' N\left(d_2(\tau, x')\right) - e^{-r\tau} K N\left(d_1(\tau, x')\right).$$

This completes the proof.

In Theorem 5 below we consider the case of a price process with a non-trivial jump part to calculate the value of the corresponding European call option.

Theorem 5 For $0 \le t < T$, the price of an European call option with respect to the neutral risk measure is given by $V(t) = c(t, S(t))$ where $S(t)$ is the price process with jumps of the underlying asset. The function $(t, x) \mapsto c(t, x)$ is defined by

$$c(t, x) = \sum_{j=0}^{\infty} \exp\left(-\tilde{\lambda}\tau\right) \frac{\tilde{\lambda}^j \tau^j}{j!} \tilde{\mathbb{E}}\left[\kappa\left(\tau, x e^{-\tilde{\lambda}\beta\varphi\tau} \prod_{i=1}^{j} (1 + \varphi Y_i)\right)\right] \quad (71)$$

with

$$\kappa(\tau, x') = x' N\left(d_2(\tau, x')\right) - e^{-r\tau} K N\left(d_1(\tau, x')\right). \quad (72)$$

The parameters κ, d_1, d_2 and τ are defined in Theorem 4 above.

Of course, in order to find the function $c(t, x)$ we combine the equalities (71) and (72) and substitute

$$x' = x e^{-\tilde{\lambda}\beta\varphi\tau} \times \prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i).$$

Proof. From the definition of the value of the option at time t, we know that the price of an European call option is given by

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)\exp\left(\sigma\left(\tilde{B}(T) - \tilde{B}(t)\right) + \psi\left(\tilde{W}(T) - \tilde{W}(t)\right) + \left(r - \tilde{\lambda}\beta\varphi - \frac{\sigma^2}{2} - \frac{\psi^2}{2}\right)(T - t)\right) \times \prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i) - K\right)^+ \vdots \mathcal{F}(t)\right]. \quad (73)$$

Since the variables $\tilde{B}(T) - \tilde{B}(t), \tilde{W}(T) - \tilde{W}(t)$ and $\tilde{N}(T) - \tilde{N}(t)$ are independent of $\mathcal{F}(t)$ and the $S(t)$ is $\mathcal{F}(t)$-measurable, from the independence Lemma 2 it follows that

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)\exp\left(\sigma\left(\tilde{B}(T) - \tilde{B}(t)\right) + \psi\left(\tilde{W}(T) - \tilde{W}(t)\right) + \left(r - \tilde{\lambda}\beta\varphi - \frac{\sigma^2}{2} - \frac{\psi^2}{2}\right)(T - t)\right) \times \prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i) - K\right)^+ \vdots \mathcal{F}(t)\right] = c\left(t, S(t)\right), \quad (74)$$

where

$$c(t, x) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)\exp\left(\sigma\left(\tilde{B}(T) - \tilde{B}(t)\right) + \psi\left(\tilde{W}(T) - \tilde{W}(t)\right) + \left(r - \tilde{\lambda}\beta\varphi - \frac{\sigma^2}{2} - \frac{\psi^2}{2}\right)(T - t)\right) \times \prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i) - K\right)^+\right]. \quad (75)$$

We rewrite the expression for $c(t, x)$ in the form

$$c(t, x) = \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)\exp\left(\sigma\left(\tilde{B}(T) - \tilde{B}(t)\right) + \psi\left(\tilde{W}(T) - \tilde{W}(t)\right) + \left(r - \tilde{\lambda}\beta\varphi - \frac{\sigma^2}{2} - \frac{\psi^2}{2}\right)(T - t)\right) \times \prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i) - K\right)^+ \vdots \sigma\left(\prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i)\right)\right]\right]. \quad (76)$$

In (76) we used the facts that the variable $\prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i)$ is measurable with respect to the σ-field $\sigma\left(\prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i)\right)$ and that the variables Y_B, Y_W do not depend on $\sigma\left(\prod_{i=1+N(t)}^{N(T)} (1 + \varphi Y_i)\right)$. Put

$$Y_B = -\frac{\tilde{B}(T) - \tilde{B}(t)}{\sqrt{\tau}}, \quad Y_W = -\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{\tau}};$$

and also $\tau = T - t$. It is clear that the random variables Y_B and Y_W are standard normal random variables with respect to the measure $\tilde{\mathbb{P}}$. By again invoking Lemma 2 from (76) and definition of κ in (54) we obtain

$$\tilde{\mathbb{E}}\left[e^{-r\tau}\left(x \exp\left(-\sigma\sqrt{\tau}Y_B - \psi\sqrt{\tau}Y_W + \left(r - \tilde{\lambda}\beta\varphi - \frac{\sigma^2}{2} - \frac{\psi^2}{2}\right)\tau\right) \times \prod_{i=1+N(t)}^{N(T)} (1\right.\right.$$

$$= \kappa\left(\tau, xe^{-\tilde{\lambda}\beta\varphi\tau} \prod_{i=1+N(t)}^{N(T)} (1+\varphi Y_i)\right). (77)$$

It follows that

$$c(t,x) = \tilde{\mathbb{E}}\left[\kappa\left(\tau, xe^{-\tilde{\lambda}\beta\varphi\tau} \prod_{i=1+N(t)}^{N(T)} (1+\varphi Y_i)\right)\right].$$

Observe that if we take the conditioning on the event $\{N(T) - N(t) = j\}, j \in \mathbb{N}$, in equality (77), the distribution of the random variables $\prod_{i=1}^{j}(1+\varphi Y_i)$ and $\prod_{i=1+N(t)}^{N(T)}(1+\varphi Y_i)$ coincide. In addition,

$$\tilde{\mathbb{P}}[N(T) - N(t) = j] = \exp(-\tilde{\lambda}\tau)\frac{\tilde{\lambda}^j \tau^j}{j!}.$$

So we obtain

$$c(t,x) = \tilde{\mathbb{E}}\left[\kappa\left(\tau, xe^{-\tilde{\lambda}\beta\varphi\tau} \prod_{i=1+N(t)}^{N(T)} (1+\varphi Y_i)\right)\right]$$

$$= \sum_{j=0}^{\infty} \exp\left(-\tilde{\lambda}(T-t)\right)\frac{\left(\tilde{\lambda}(T-t)\right)^j}{j!} \times$$

$$\left[x \exp\left(-\tilde{\lambda}\beta\varphi(T-t)\right)(1+\varphi\beta)^j \tilde{\mathbb{E}}\left(N(d_2^j(T-t,x))\right) - \exp\left(-r(T_t)\right)K\tilde{\mathbb{E}}\left(N(d_1^j(T-t,x))\right)\right]. \qquad (78)$$

Because

$$\kappa(\tau,x') = x'N\left(d_2(\tau,x')\right) - e^{-r\tau}KN\left(d_1(\tau,x')\right).$$

This completes the proof of Theorem

5. Application

Here we are going to consider a real example about an investment in oil sector [4]. We will do a simulation of the asset price process and we will see its behavior at each time t. In proposition 1, the asset price process is given below.

$$S(t) = S(0)\exp\left[\int_0^t \left(b(s) - \lambda\beta\varphi(s)\right)ds + \int_0^t \sigma(s)\,dB(s) + \int_0^t \psi(s)\,dW(s)\right] \times$$

$$\exp\left[-\frac{1}{2}\int_0^t \left(\sigma^2(s) + \psi^2(s)\right)ds + \sum_{0<s\leq t} \ln\left(1+\varphi(s)\Delta Q(s)\right)\right] \quad (96)$$

We used GNU Octave version 3.8.1 for doing simulation. The simulation gives us two curves, which include the one for the asset price process with jumps and the one of the asset price process without jumps. To obtain this plot, Figure 1 below, we setted that $b(s), \lambda, \beta, \sigma(s), \psi$ and $\varphi(s)$ are constants. Letting that $b(s) = 0.26, \lambda = 0.05, \beta = 0.05, \sigma(s) = 0.25$. The one with without jumps is obtained by setting that $\varphi(s) = 0$ in the expression of $S(t)$, Expression (96).

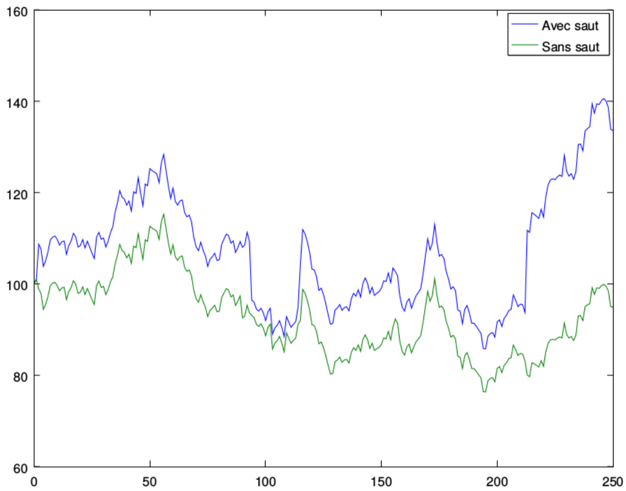

Figure 1: Evolution of price process with jumps and without jumps

6. Concluding remarks

In the terminology of this paper, we formulated the problem of inertia through the price process of the underlying risky asset of which the evolution is described by stochastic differential equation with jumps. Compared to the general form of this type of equations (i.e. stochastic differential equations with jumps), the stochastic differential equation in a model with inertia possesses an extra Brownian term. This term contains the parameter of the disequilibrium of the market caused by the inertial behavior of the small invest agents.

By stochastic calculus we have determined the price of an European call option in an explicit manner for a financial market with inertia. We also established that the price of an European call option is a solution to an integro-differential equation which depends on the disequilibrium parameter of the financial market.

Finally, we have done with a numerical example about the asset price evolution; both considering the evolution of asset price with jumps and without jumps. In the future, we can consider the evolution of option price with jumps and without jumps.

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