Exploring the Mathematical Properties of and Related to the Tower of Hanoi and Its Solution

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Abstract: ***The Tower of Hanoi is arguably one of the most fascinating mathematical puzzles of all time. Apparently based on the legend of a mystical Hindu temple where 64 golden discs were placed in increasing size from top to bottom, the modern version introduced by Edouard Lucas has fascinated many math-enthusiasts worldwide. Only one rule must be followed while moving all discs from the first to the last column in order to complete the puzzle – a larger disc must never be placed over a smaller one. The puzzle has encouraged countless explorations of the logic and the mathematics behind the puzzle, with more and more striking patterns becoming apparent over time. This paper will explore a basic solution of the puzzle, followed by a mathematical and combinatorial derivation of formulae relating to the minimum and maximum number of moves that can be used to solve any version of the puzzle (with a specified number of discs and columns). Some variations and applications of the puzzle and its related concepts will also be explored.***

Keywords: Tower of Hanoi, Lucas Tower, Tower of Brahma, Mathematical Puzzle, Mathematics

1. Introduction

The Tower of Hanoi is a mathematical puzzle that was introduced by French mathematician Edouard Lucas in 1883. It is also called the Tower of Brahma or the Lucas Tower. The concept of the puzzle is believed to be based on the legend of a Hindu temple, where 64 discs made of gold, of increasing size, were placed top to bottom on one of three posts. The priests were given the task of moving all the discs on to another post, following the rule that a larger disc could never be placed over a smaller disc, as the weight of the larger would crush the smaller one. It was said that when the priests completed the task, the world would end.

The general Lucas Tower consists of three pegs, although variations of the puzzle have been developed to include a greater number of pegs to increase the complexity. Most modern versions of the puzzle usually use between 3 and 8 discs, but the original puzzle was created to have anywhere between 1 and 64 discs. Varied solutions have been developed over time, with more and more mathematical patterns becoming apparent.

The rules of solving the puzzle include that no larger disc can be placed over a smaller one, only the top disc from a peg can be moved, and discs can be moved only one at a time.

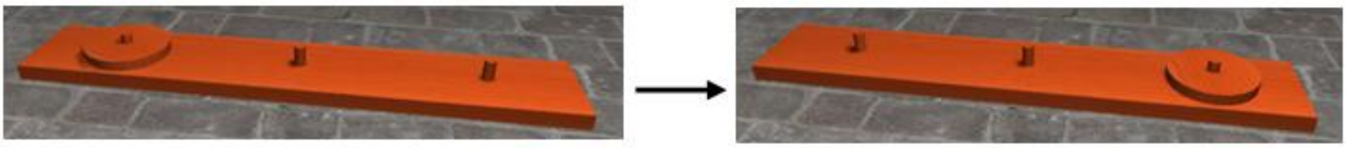
2. Aim

The exploration aims to investigate solutions starting with the basic versions of the Lucas Tower and generalise a recursive expression for the number of moves required to complete the puzzle. Further, it aims to conjecture and prove a general expression for the number of moves required to solve the puzzle, and then explore applications and other variations of the puzzle, along with their solutions.

Basic Versions of the Lucas Tower and their solutions:

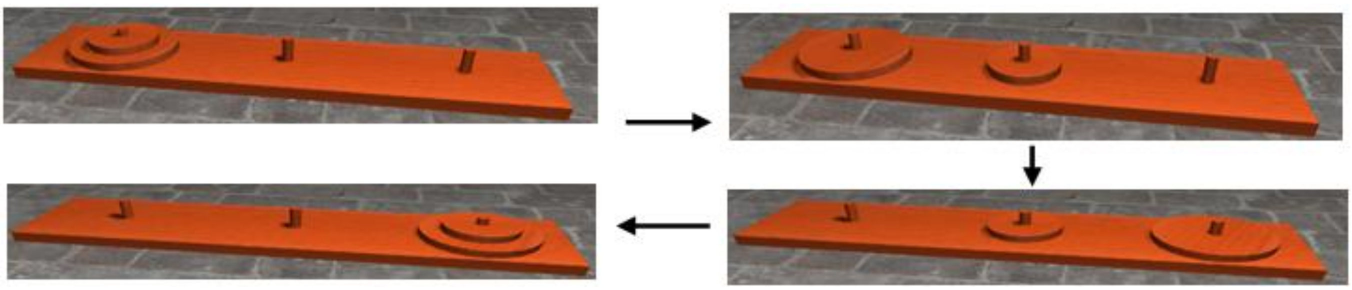
We will first explore solutions to the basic versions of the Tower of Hanoi involving three pegs. In the steps below, discs are numbered starting 1, 2 and so on from top to bottom and the pegs are A, B, and C from left to right.

1 disc, 3 pegs:



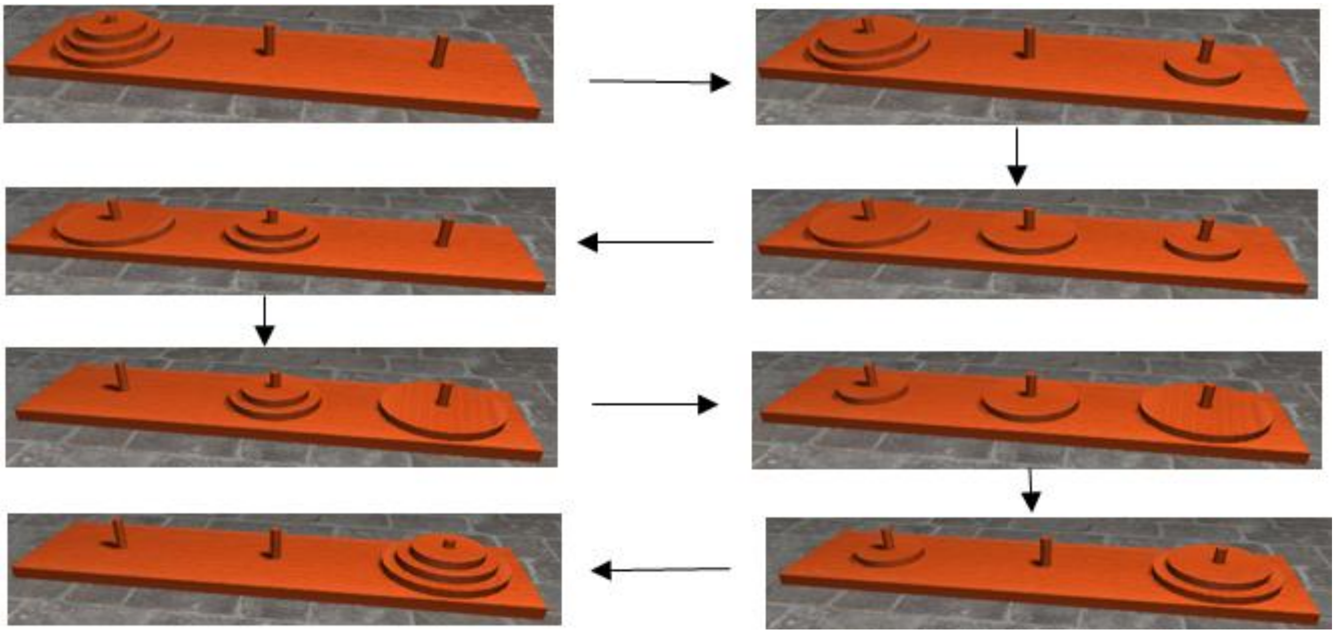
1.) Move disc 1 to the right from peg A to C
Minimum number of moves required: 1

2 discs, 3 pegs:



1.) Move disc 1 to the right from peg A to B 2.) Move disc 2 to the right from peg A to C
3.) Move disc 1 to the right from peg B to C
Minimum number of moves required: 3

3 discs, 3 pegs:



1.) Move disc 1 to the right from peg A to C2.) Move disc 2 to the right from peg A to B
3.) Move disc 1 to the left from peg C to B 4.) Move disc 3 to the right from peg A to C
5.) Move disc 1 to the left from peg B to A 6.) Move disc 2 to the right from peg B to C
7.) Move disc 1 to the right from peg A to C
Minimum number of moves required: 7

By further solving the puzzle using 4 and 5 discs, we observe that the minimum numbers of moves required are 15 and 31, respectively. Upon solving the puzzle with an increasing number of discs, a recursive pattern seems to be emerging. For example, in the case of 3 discs, we first shifted the top 2 discs to peg B and then after shifting disc 3 to peg C, we shifted discs 1 and 2 from peg B to C. A similar pattern was repeated for 4 and 5 discs.

Recursive solution and formula:

Essentially, for any number of discs, we initially ignore the bottom disc placed at peg A and move all the other discs on top to peg B. Then, we move the largest disc in one move from A to C, followed by the remaining discs being moved from peg B to C. So, for any N number of discs, we first shift the top N-1 discs to the second peg. Ignoring the presence of the N^{th} disc, this is equivalent to solving the puzzle for N-1 discs and would require the same number of moves. Then, a single move shifts the N^{th} disc to the target peg and subsequently the remaining N-1 discs on the centre peg are shifted to the target peg (once again, ignoring the presence of the N^{th} disc on the target peg, this is equivalent to solving the puzzle for N-1 discs). Therefore, solving the puzzle for N discs entails solving the puzzle twice for N-1 discs plus a single additional move.

Let $\mathbf{MIN_N}$ be the minimum number of moves required to solve the Lucas Tower involving N discs, then by the recursive logic explained above:

$$\mathbf{MIN_{N+1} = MIN_N + 1 + MIN_N = 2MIN_N + 1}$$

This relation matches the number of moves required to solve the puzzle until 5 discs, as verified using the online simulator.

However, to apply the above relation to N number of discs, we first need to have knowledge of the required number of moves for N-1 discs. Using the "Recursion" function on the CASIO fx-CG50 and entering the expression "$a_{n+1} = 2a_n + 1$", we can obtain the approximate number of moves required for N discs ($1 \leq N \leq 64$) where the number of moves for 64 discs is approximately 1.8×10^{19} moves. So $MIN_{64} \approx 1.8 \times 10^{19}$

However, these methods are not viable for a larger number of discs and hence let us try to conjecture a formula for MIN_N solely in terms of N.

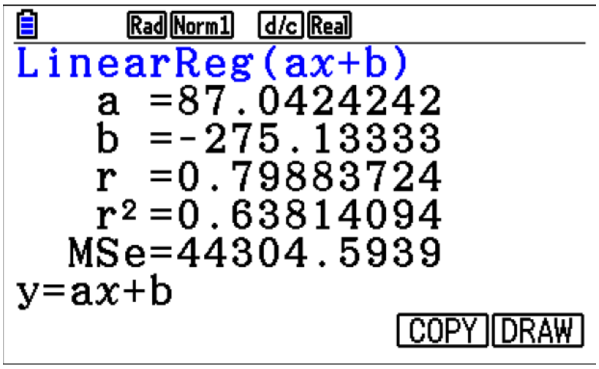
Conjecturing a formula for $\mathbf{MIN_N}$ in terms of N:

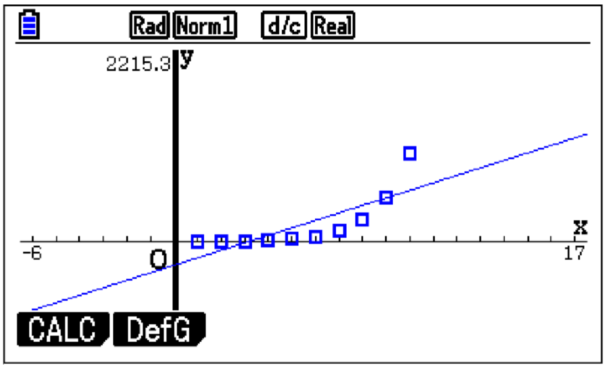
Using the recursion formula, we first calculate MIN_N until N=10and then try to approximate a relationship between the two using the statistical functions on the fx-CG50 as well as Microsoft Excel.

The table below gives MIN_N until N=10.

N	MIN_N	N	MIN_N
1	1	6	63
2	3	7	127
3	7	8	255
4	15	9	511
5	31	10	1023

First, let us try to relate N to MIN_N using a linear relation of the form y = ax + b.

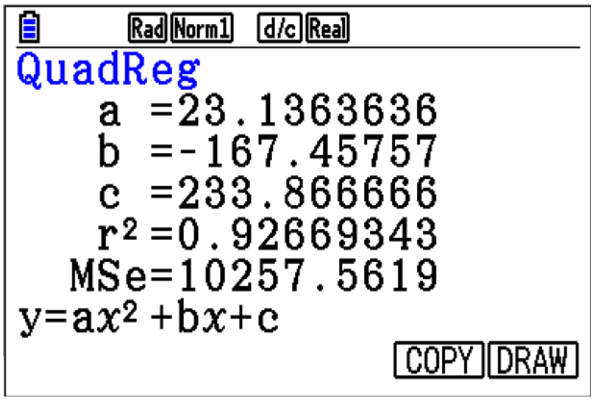


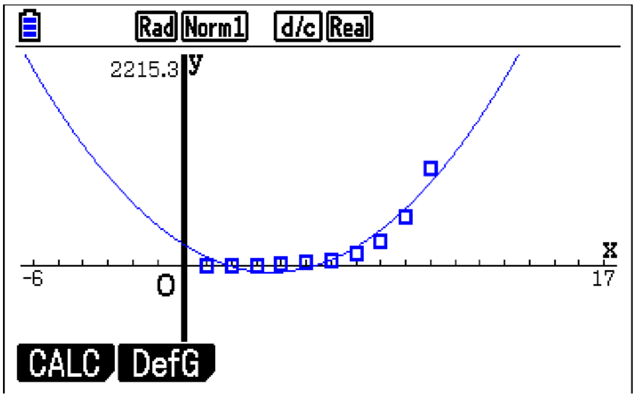


The above linear regression data (obtaining using the "Statistics" function on the fx-CG50) indicates that a linear relationship can account for only ≈63.8% of the relationship between the dependent variable (MIN_N) and the independent variable (N), as indicated by the "r^2" value. Further, the evident discrepancy between the predicted line and the true values indicates that a linear the relationship is appropriate.

Although the mean squared error (MSe) value is usually a good indicator of the extent to which the modelled relationship fits the true values, it may not be too useful as an indicator with the current data due to large increases in the dependent variable as the independent variable increases.

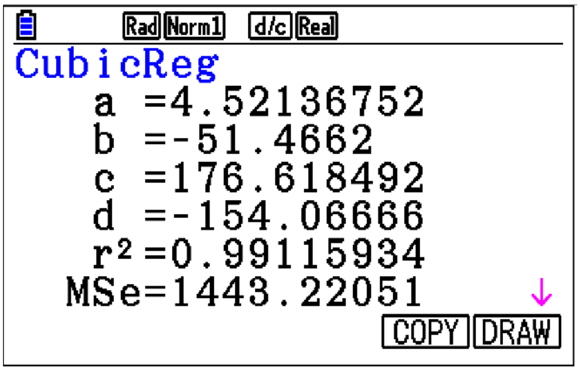
Next, let us try modelling the data using polynomial relations.

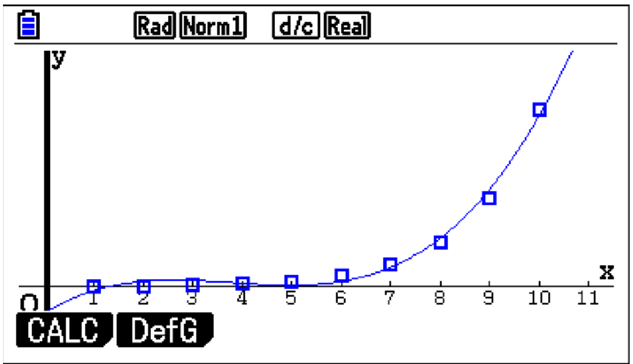




The quadratic relation seems to be a closer fit than that the linear relation. It accounts for ≈92.7% of the relationship between N and MIN_N. The plotted graph is closer to the true data points as well.

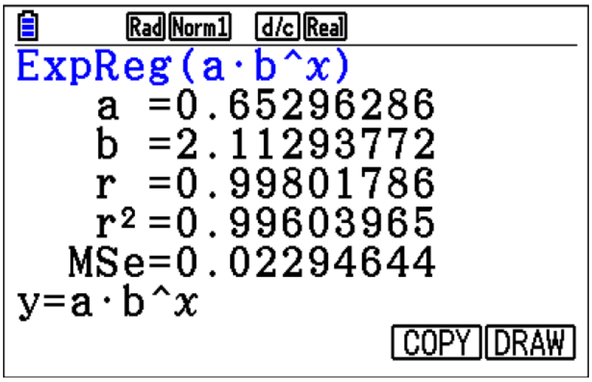
We should also try using higher order polynomials:

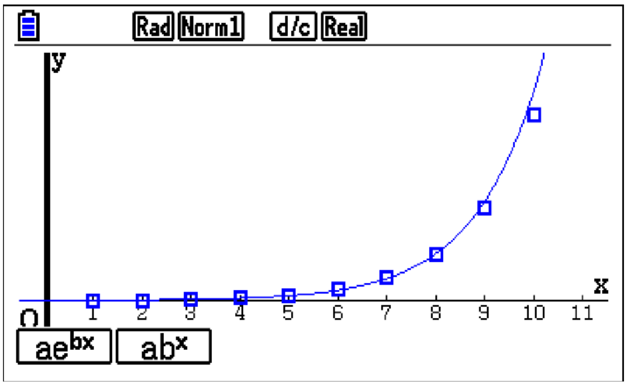




As can be seen above, the cubic relation is a much closer fit to the data, seemingly representing ≈99.1% of the relationship between the two variables. On further using higher order relations, (4, 5, and 6) we obtain higher values for "r^2" almost close to 1, i.e., seemingly accounting for almost 100% of the relationship.

Despite the apparent good fit of the high order polynomial functions, we should try an exponential relation as well given the nature of the plotted data.





The exponential relation seems to consider ≈99.6% of the relation between N and MIN_N. Although this value may be less than that for the higher order polynomials (5 and 6), the exponential relation is simpler, its graph almost perfectly matches the plotted data points, and intuitively seems to be a better fit for the values. Further, the high r-squared values for the higher order polynomials may simply be due to them being overfit models (this occurs when more predictors are added by the use of higher order polynomials and the result in models that are too complicated for the data and yield misleading high values for r^2).

The relation is ≈ 0.65×2.11^xwhich led to a further investigation concerning the exponential powers of 2. The observations are given in the table below:

N	MIN_N	2^N	N	MIN_N	2^N
1	1	2	6	63	64
2	3	4	7	127	128
3	7	8	8	255	256
4	15	16	9	511	512
5	31	32	10	1023	1024

According to the table above, MIN_N appears to be one less than 2^N.

Conjecture: $MIN_N = 2^N - 1$

Proving the formula for MIN_N in terms of N:

We shall try to prove the conjecture using mathematical induction.

To prove: $MIN_N = 2^N - 1$; where $N \in \mathbb{N}$

For N=1, the minimum moves required to solve the puzzle is clearly equal to 1.

So, for N=1, $MIN_N = 2^N - 1 = 2^1 - 1 = 1$

Hence, the formula is true for N=1

Let us assume that the expression is true for a number k

$MIN_k = 2^k - 1$

Now, using the recursive formula $MIN_{N+1} = 2MIN_N + 1$:

$MIN_{k+1} = 2MIN_k + 1 = 2 \times (2^k - 1) + 1 = 2\times 2^k - 2 + 1 = 2^{1+k} - 1 = 2^{k+1} - 1$

Therefore, if MIN_k is true, MIN_{k+1} is also true.

But we have already shown that $MIN_N = 2^N - 1$for N=1

Hence, by induction:

MIN_1 is true, so MIN_2 is also true.

MIN_2 is true, so MIN_3 is also true and so on until:

Since MIN_{N-1} is true, MIN_N is also true.

Therefore, using Mathematical Induction, we can prove that $MIN_N = 2^N - 1$, i.e., the minimum number of moves required to solve a Lucas Tower with N discs is $2^N - 1$.

Application and features of the formula:

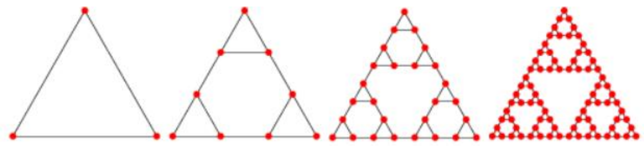
Using the formula for N=64, i.e., the original myth, we obtain that the minimum number of moves required is $2^{64} - 1 \approx 1.844674407 \times 10^{19}$ moves (this is similar to the approximation estimated using the recursion formula).

If the priests in the original myth were to make one move per second, it would take them $1.844674407 \times 10^{19}$ seconds, i.e., approximately 585 billion years to complete the task until the world ends.

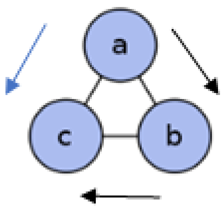
Another interesting connection is that the formula for MIN_N is also the formula for the n^{th} Mersenne number. Numbers which are one less than a power of 2 are called Mersenne numbers, named after Marin Mersenne. Mersenne primes are prime numbers of the form $2^n - 1$. Edouard Lucas, in fact, developed a test for Mersenne numbers to check whether they were prime or not, called the Lucas-Lehmer primality test. Lucas also proved that 2^{127}-1 was a prime, which is, till date, the largest prime found without the use of a computer. His work in Mersenne numbers and primes is a possible inspiration for his introduction of the Tower of Hanoi puzzle.

Hanoi Graphs and their application in the context of the puzzle:

A Hanoi graph is an undirected graph (its edges can be traversed in either direction) where each of its nodes represents one of the possible configurations of the position of the discs on the pegs in the Tower of Hanoi puzzle. Assuming the usual case of 3 pegs, a Hanoi graph for n discs will have 3^nnodes because each configuration is represented by choosing one tower for each disc. Given below, from left to right, are the Hanoi graphs for 1, 2, 3, and 4 discs respectively.



Every Hanoi graph has one Hamiltonian cycle (a circuit through a graph that passes each node exactly once). Hence a Hanoi graph is a Hamiltonian graph (it possesses a Hamiltonian cycle). In the case of 3 pegs and 1 disc, the Hanoi graph is given below.



As shown by the graph above, there are three possible positions for the one disc, on pegs A, B, or C. Assuming A is the first peg on the left and C is the target peg, the shortest path with minimum moves would be the one directly from A to C depicted by the blue line. But the longest path with the maximum number of moves, without the repetition of a position, would be from A to B and then to C, depicted by the black arrows. This is essentially the graph's Hamiltonian cycle. As evident, the longest path took one less move than the number of vertices on the graph. This is because you cannot repeat a position and to move from any node x to another node y, where the total number of nodes in the path including x and y are 'p', you would need 'p-1' moves. So, to move from A to C via B takes 3-1 = 2 moves.

To solve the Tower of Hanoi using the maximum number of moves without repeating any position, you would have to pass every node in the Hanoi graph for n number of discs, i.e.,3^n nodes. But, as discussed above, the number of moves this would take would be one less than the total number of nodes in the graph, i.e.,$3^n - 1$ moves. Solving the Tower of Hanoi with the maximum number of moves possible would take $\mathbf{3^n - 1}$moves. For the case of 64 discs, this would mean that it would take the priests $3^{64} - 1$ seconds $\approx 3.43368 \times 10^{30}$ seconds or approximately 1.1×10^{23} years if they used the longest method to complete the task.

Solving the puzzle using the above method is akin to solving the puzzle with the least number of moves incorporating the modification that discs can only be moved between adjacent pegs (a disc cannot be moved directly from A to C or vice versa but must be moved from A to B and then to C). This would imply that each possible configuration of positions on the Hanoi graph would be used and $3^n - 1$ moves would be required. The halfway mark will be reached (shifting all discs from A to B) using exactly $\frac{3^n-1}{2}$ moves, i.e., the puzzle would essentially be divided into two equal parts.

Variations of the Tower of Hanoi:

The puzzle becomes significantly more difficult when the pegs are increase to a number beyond three. Originally the puzzle using 4 pegs was called the Reve's puzzle. An algorithm called the Frame-Stewart algorithm has been commonly used to solve this puzzle, as well as other variations of the Tower of Hanoi with more pegs.

Let the minimum number of moves required to complete the puzzle with p pegs and n discs be MOV(p,n). The general methodology of the algorithm is as follows:

→ for a number of discs x where $1 \leq x < n$, transfer the first x discs from the top of the initial peg to a peg in between the initial and target pegs by MOV(p,x) moves
→ the other n-x discs on the initial peg must be shifted to the target peg by making use of p-1 pegs (the peg with the first x discs must not be used) and MOV(p-1,n-x) moves
→ now, the presence of the n-x discs on the target peg can be ignored, and the remaining x discs should be shifted to the target peg using MOV(p,x)

In this way, the Frame-Stewart algorithm is supposed to take 2MOV(p,x) + MOV(p-1,n-x) moves. The crucial element is choosing a suitable value of x which could make the solution optimal and use the minimum number of moves. Although the algorithm has neither been proved nor disproved, it has proven to be the most optimal until 30 discs in the variation with 4 pegs (Reve's puzzle).

3. Applications

The Tower of Hanoi has long been used for psychological experimentation and research. It has also been used widely in computational and recursion programming. It is also applied as a backup rotation scheme in cases where multiple storage devices and systems are used.

4. Conclusion

This investigation was able to successfully obtain the recursion formula for the recursive solution to the original version of the Lucas Tower. Using the statistical functions on the fx CG-50 and Microsoft Excel, we were able to conjecture a formula for the minimum number of moves required to solve the puzzle and further prove it using mathematical induction. Applying the formula yielded the number of moves it would take to solve the 64-disc puzzle and to truly appreciate the large number we expressed it in terms of the amount of time it would take assuming one move was made per second (this was a relatively quick rate to assume but this was done for simplicity). Using Hanoi graphs, we were able to appreciate a graphical representation of the various possible configurations in the process of solving the Lucas Tower and also derive an expression for the maximum number of moves in which the puzzle could be solved. To appreciate the difference between the two paths for a larger number of discs, we calculated the approximate time it would take to solve the 64-disc puzzle using the maximum moves. We further explored variations of the Lucas Tower, including the Reve's puzzle and the algorithm which is widely used to solve it. The various applications of the Tower of Hanoi and its concepts were also discussed.

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