Effects of Mach Number and Prandtl Number on Plane Couette Flow

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Abstract: ***If the flow is compressible, heat due to friction as well as temperature change due to compression must be taken into account. In addition, it is necessary to consider the effects of the variation of viscosity with temperature. This dependence of temperature with viscosity and density makes the calculation of problems in viscous compressible flow much more difficult than that for the case of in-compressible flow. It's necessary to consider equation of continuity momentum energy and reduced Navier-stokes equation to the the phenomenon of plane couette-flow for different values of Mach number and Prandtl numbers. The significance of four important Controlling parameters in viscous compressible fluids need to be discussed and illustrated.***

Keywords: Navier-stokes equation, Mach number, prandtl number, temperature, velocity, viscosity, thermal conductivity, ratio of specific heats

Nomenclature

ρ−density
R−gas constant
p−pressure
τw−wall shearing stress
u−velocity component along x axis
Pr−prandtl number
v−velocity component along y axis
M−Mach number
w−velocity component along z axis
m−mass
q −velocity vector
M_∞−Mach number of the plate motion
μ−coefficient of viscosity
T∞−temperature of moving plate
k−thermal conductivity
U−constant velocity of moving plate
T−temperature
qw−heat flow to the wall
p−specific heat at constant pressure
k−ratio of specific heats
h−distance between the plates
T_r−recovery temperature

1. Introduction

It may be recalled that a Couette flow is the flow between two parallel flat plates, the lower plate is at rest and the upper plate is moving with a velocity U parallel to the fixed plate.

For a viscous compressible fluid in steady two-dimensional flow the Navier-Stokes equations [1, 2, 3] in Cartesian coordinates with negligible body forces can be reduced to yield.

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-2\left(\nabla .q\right)/3\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \quad (1.1)$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-2\left(\nabla .q\right)/3\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \quad (1.2)$$

The continuity equation is

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \quad (1.3)$$

The energy equation is reduced to

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$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\frac{2}{3}\mu(\nabla .q)^2+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2= \rho u\frac{\partial C_pT}{\partial x}+\rho v\frac{\partial C_pT}{\partial y}-\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right) \quad (1.4)$$

The equation of state for a perfect gas is

$p = \rho RT$ (1.5)

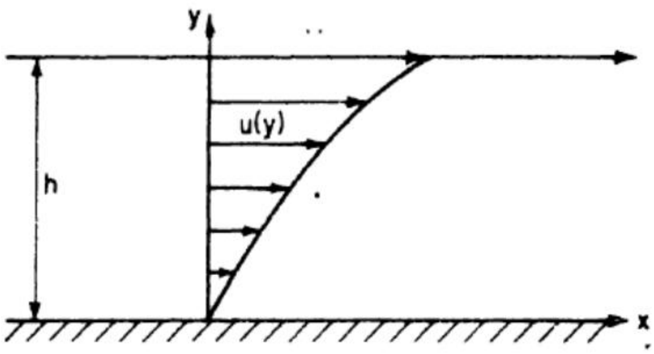
Let x be the direction of the flow, y the direction normal to the flow, and the width of the plates parallel to the z-direction be large compared to the distance h between the plates. For this simple configuration, we have

$$v=0, \quad w=0, \quad \frac{\partial}{\partial z}=0, \quad p=constant \quad (1.6)$$

The boundary conditions are then

$y = 0;\ u = 0;\ v = 0;\ q = q_w;\ (\tau_{yz}) = \tau_w$

$y = h;\ u = U;\ v = 0;\ p = p_1;$ (1.7)



According to Equations (1.3) and (1.7) and the velocity component u is independent of x. Hence, Equations 1.1 and 1.2 and reduce to

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right)=0 \quad (1.8)$$

which yields, after integration,

$$\mu\frac{du}{dy}=Constant \quad (1.9)$$

The component of the velocity u is found,

$$u=\tau_w\int_0^y\frac{dy}{\mu} \quad (1.10)$$

where the constant of integration τw. is determined from the condition at the fixed plate as

$$\tau_w=\mu_w\left(\frac{du}{dy}\right)_w=\mu\frac{du}{dy} \quad (1.11)$$

The energy equation under the present flow conditions reduces to (T∼1/ρ ∼ u),

$$\frac{d}{dy}\left(k\frac{dT}{dy}\right)=-\mu\left(\frac{du}{dy}\right)^2 \quad (1.12)$$

Substituting Equation (1.11) into (1.12), we obtain

$$\frac{d}{dy}\left(k\frac{dT}{dy}\right)=-\frac{\tau_w^2}{\mu} \quad (1.13)$$

Integrating, we get

$$k\left(\frac{dT}{dy}\right)=-\tau_w^2\int_0^y\frac{dy}{\mu}+C=-\tau_w u+C=-\mu u\frac{du}{dy}+C \quad (1.14)$$

where the constant of integration is determined from the condition at the fixed plate, i.e.,

$$C=k_w\left(\frac{dT}{dy}\right)_w=-q_w \quad (1.15)$$

The negative sign is conventional, to show that q_w, (the flow of heat [2] through the wall per unit time) is positive when $(dT/dy)_w$, is negative. With the relation in Equation 1.15 we can rewrite Equation 1.16 in the form

$$k\left(\frac{dT}{dy}\right)+\mu u\frac{du}{dy}=-q_w \quad (1.16)$$

From either the simple kinetic theory of gases or empirical data, the coefficient of viscosity μ can often be expressed with sufficient accuracy as a power of the absolute temperature,

$$\frac{\mu}{\mu_\infty}=\left(\frac{T}{T_\infty}\right)^m \quad (1.17)$$

For air at ordinary temperature, m = 0:76 is generally used. As the temperature increases, m decreases toward 1/2. The Prandtl number Pr is very nearly constant (of order unity) for all common gases. Since C_p is also nearly constant for a fairly wide range of temperatures around ordinary temperatures, the coefficient of heat conductivity [3] k is directly proportional to μ. Based on these arguments Eq. (12-9) may be written as

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$$\mu\frac{d}{dy}\left(\frac{C_p}{Pr}T+u^2/2\right)=-q_w \quad (1.18)$$

The integration of Equation (1.15) results in

$$C_pT+Pru^2/2=-Prq_w\int_o^v\frac{1}{\mu}dy+C \quad (1.19)$$

where $C = C_pT_w$ and T_w corresponds to the temperature at the fixed plate. Substituting equation (1.10) into equation (1.19), one finds

$$C_pT+Pru^2/2=-Pr\frac{q_wu}{\tau_w}+C \quad (1.20)$$

2. Temperature and Velocity Distributions in Couette flow

If the temperature of the moving plate is denoted by $T\infty$ the constant of integration in Equation (1.20) may be determined from the boundary condition at the upper plate, i.e.,

$$C=C_pT_\infty+\frac{1}{2}PrU^2+PrU\frac{q_w}{\tau_w} \quad (2.1)$$

With this value of C, equation (1.20) becomes

$$C_p\left(T-T_\infty\right)=Pr(U^2-u^2)/2+Pr\frac{q_w(U-u)}{\tau_w}+C \quad (2.2)$$

Dividing the above equation by C_pT_∞. and remembering

$$u^2/C_pT_\infty=(k—1)M_\infty^2$$

the temperature distribution in Couette flow is obtained:

$$\left(\frac{T}{T_\infty}\right)=1+\frac{Pr}{2}(k-1)M_\infty^2(1-\frac{u^2}{U^2})+Pr\frac{q_w(U-u)}{\tau_w}M_\infty\left(1-\frac{u}{U}\right)+C \quad (2.3)$$

The velocity distribution in Couette flow may be obtained as follows

$$\tau_wy=\int_o^u\mu du \quad (2.4)$$

With the relation of μ in equation (1.17) and T in (2.3), (2.4) may be written as

$$\left(\frac{\tau_wy}{\mu_\infty}\right)=\int_0^u\left(1+\frac{Pr}{2}(k-1)M_\infty^2(1-\frac{u^2}{U^2})+Pr\frac{q_w(U-u)}{\tau_w}M_\infty\left(1-\frac{u}{U}\right)\right)^m du \quad (2.5)$$

For an arbitrary value of m the integrals have to be evaluated numerically. If m = 1 equation (2.5) yields

$$\left(\frac{\tau_wy}{\mu_\infty U}\right)=\frac{u}{U}+\frac{Pr}{2}(k-1)M_\infty^2(\frac{u}{U}-\frac{u^3}{3U^3})+Pr\frac{q_w(k-1)}{\tau_wU}M_\infty^2\left(\frac{u}{U}-\frac{1}{2}\left(\frac{u}{U}\right)^2\right) \quad (2.6)$$

For the case of an adiabatic wall, $q_w = 0$, and (2.6) reduces to

$$\left(\frac{\tau_wy}{\mu_\infty U}\right)=\frac{u}{U}+\frac{Pr}{2}(k-1)M_\infty^2(\frac{u}{U}-\frac{u^3}{3U^3}) \quad (2.7)$$

The shearing stress T_w can be obtained by letting u = U and y = h. The result may be written as

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$$\frac{\tau_w y}{\mu_\infty U} = 1 + \frac{Pr}{3}(k-1)M_\infty^2 \quad (2.8)$$

Dividing (2.7) by (2.8), we obtain

$$\frac{y}{h} = \frac{1}{1+\frac{Pr}{3}(k-1)M_\infty^2}\left(\frac{u}{U} + \frac{Pr}{2}(k-1)M_\infty^2\left(\frac{u}{U} - \frac{u^3}{3U^3}\right)\right) \quad (2.9)$$

When

$$M_\infty^2 \to \infty$$

we have

$$\frac{y}{h} = \frac{3u}{2U}\left(1 - \frac{u^3}{U^3}\right) \quad (2.10)$$

The velocity distribution of the Couette flow is plotted in Fig.(2.1) as a function of the distance from the fixed wall for various Mach numbers and Prandtl numbers[7]. It is seen that the effect of the Mach number on the Couette flow is to decrease the velocity gradient at the stationary wall and to increase it at the moving wall. Since the Prandtl number appears paired with the Mach number (both increase heat transfer to the fluid), it has the same influence on the velocity distribution as the Mach number. A comparison of the velocity distributions for m = 1:0 and m = 0:76 also depicted in Fig. (2.1).

The temperature distribution of the Couette flow can be easily calculated [7] once the velocity distribution has been determined.

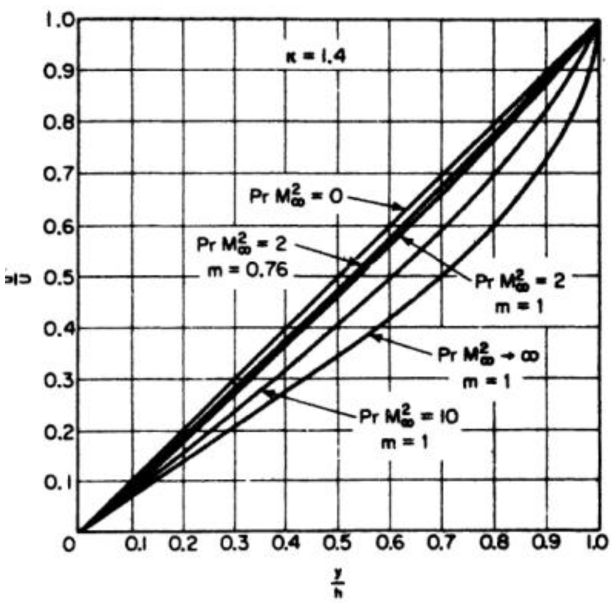

Figure 2.1

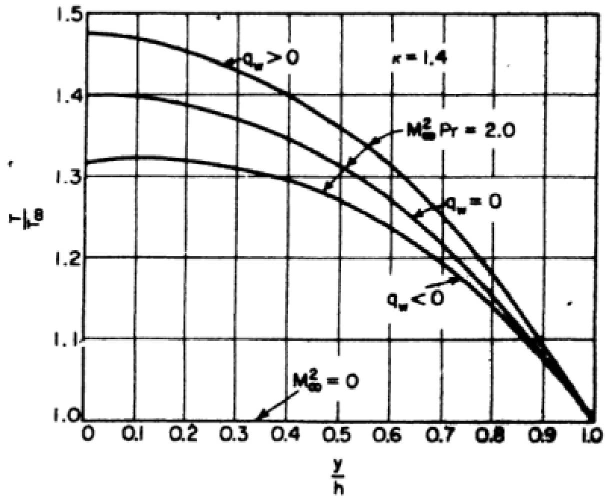

Figure 2.2

Typical temperature profiles in Couette flow ($PrM^1_\infty = 2$) for an adiabatic wall ($q_w = 0$), a heated wall ($q_w < 0$), and a cooled wall ($q_w > 0$) are shown in Fig. (2.3). It is interesting to note that the temperature remains unchanged for an incompressible fluid. Also, the temperature gradient becomes zero for a compressible flow with an adiabatic wall. This special temperature at the fixed insulated wall (adiabatic wall) is called the recovery temperature, and is denoted by T_r. According to (2.3) we can find T_r, by putting $q_w = 0$ and $u = 0$ as follows:

$$\frac{T_r}{T_\infty} = 1 + \frac{Pr}{2}(k-1)M_\infty^2 \quad (2.11)$$

The coefficient of friction [8] at the fixed wall ($q_w = 0$) may be obtained from (2.8),

$$C_r = \frac{\tau_w}{\rho_\infty(U^2/2)} = \frac{1+\frac{Pr}{3}(k-1)M_\infty^2}{Re/2} \quad (2.12)$$

where Re = Uh=v_∞. The shearing stress of (2.7) reduces to the first term when $M\infty \to 0$.

It must be noted that the velocity gradient for a compressible fluid varies from the stationary wall to the

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moving wall Fig. (2.1), and the shearing stress given in (2.7) is a constant in Couette flow. The constancy of the shearing stress in Couette flow [8, 9, 10] can be shown from the velocity gradient by

$$\tau_w = \mu_w \left(\frac{du}{dy}\right)_w = \mu_\infty \left(\frac{du}{dy}\right)_\infty \quad (2.13)$$

According to this expression we need only to show that

$$\frac{\mu_w}{\mu_\infty} = \left(\frac{du}{dy}\right)_w / \left(\frac{du}{dy}\right)_\infty \quad (2.14)$$

The viscosity is given by (1.17), and with the aid of (2.3) it may be written as ($q_w = 0$)

$$\frac{\mu_w}{\mu_\infty} = \frac{T_w}{T_\infty} = 1 + \frac{Pr}{2}(k-1)M_\infty^2 \quad (2.15)$$

Differentiating (2.7) with respect to y and imposing the limits y = 0 and y = h, we have

$$\left(\frac{du}{dy}\right)_w = \frac{\tau_w}{\mu_\infty} \frac{1}{1 + \frac{Pr}{2}(k-1)M_\infty^2} \quad (2.16)$$

and

$$\left(\frac{du}{dy}\right)_w = \frac{\tau_w}{\mu_\infty} \quad (2.17)$$

The ratio of the velocity gradients in (2.15) and (2.16) will give

$$\frac{\left(\frac{du}{dy}\right)_\infty}{\left(\frac{du}{dy}\right)_w} = 1 + \frac{Pr}{2}(k-1)M_\infty^2 \quad (2.18)$$

Hence (2.15) is verified and the shearing stress is constant in Couette flow.

The increase of the skin-friction coefficient at the fixed wall ($q_w = 0$) with the increase of the Mach number and Prandtl number is depicted in below Fig. (2.3)

3. Conclusions

Plane couette flow is discussed. The effects of mach number and prandtl number of the velocity distribution and coefficient of friction are shown with figures wherever necessary. Temperature distributions and heat flux at the wall are illustrated for different values of mach number and Prandtl number.

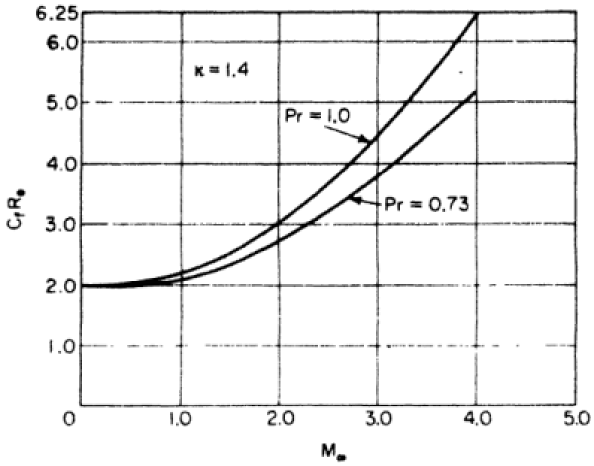

Figure 2.3

4. Scope of Research

The observations that are made in this paper plays vital role in problems related to mass and Heat transfer flows and Aerodynamics.

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